Hochschule München University of Applied Sciences

Laboratory for Mechatronic and Renewable Energy Systems (LMRES)

Optimal operation of synchronous machines – Part I

A complete framework to minimize copper & iron losses

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Outline

Outline

Motivation

Synchronous machines: Widely-used, compact and efficient actuators

www.bmw.de

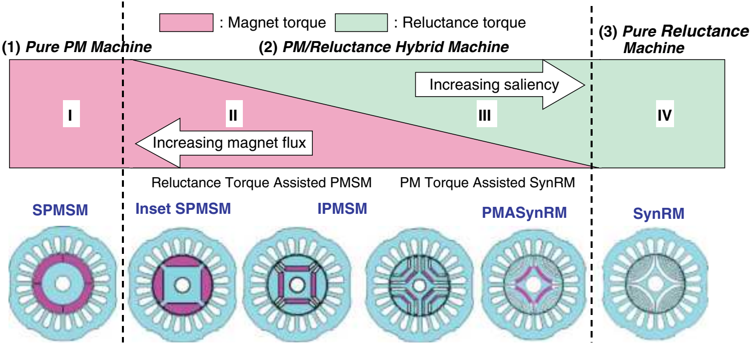
www.farmwood.co.uk

www.abb.com (from video)

- traction drives (e.g. electric vehicles or trains)
- air conditioning/fans
- pumps
- conventional power plants (e.g. hydro power plants)
- renewable energy systems (e.g. direct-drive wind turbine systems)

Motivation

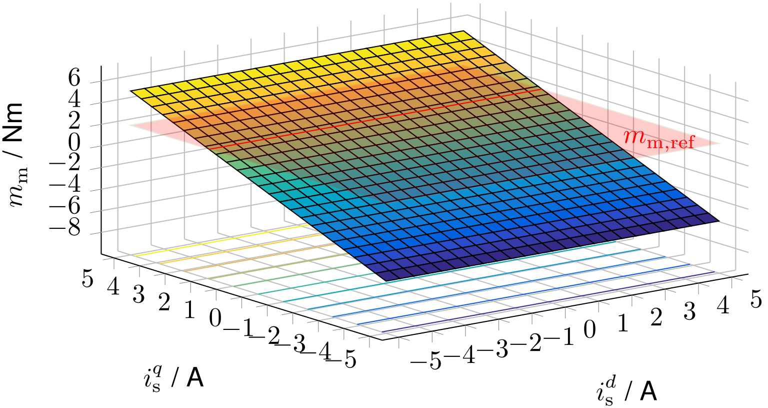
Examples of anisotropic synchronous machines [1] with "saliency ratio" $L_s^d / L_s^q \neq 1$



$$m_m(i_s^d, i_s^q) = \frac{3}{2} n_p \Big[\underbrace{\psi_{pm} i_s^q}_{\text{magnetic torque}} + \underbrace{(L_s^d - L_s^q) i_s^d i_s^q}_{\text{reluctance torque}} + \underbrace{L_{s,m}\big((i_s^q)^2 - (i_s^d)^2\big)}_{\text{cross-coupling torque}} \Big]$$

Motivation

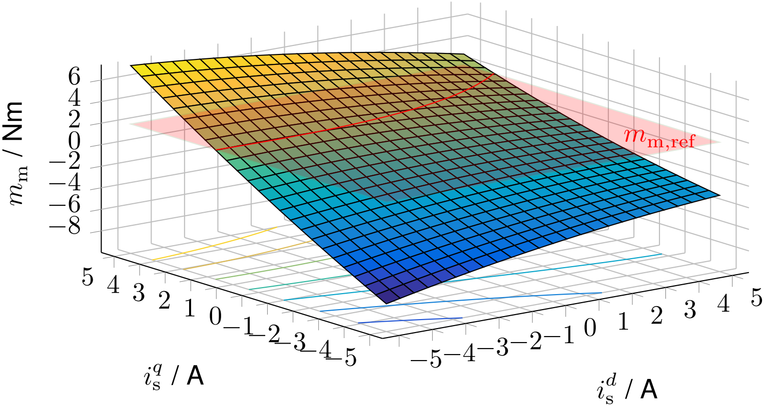
Optimal feedforward torque control problem: **Isotropic machine**



$$m_{\mathrm{m,ref}} \overset{!}{=} m_{\mathrm{m}}\left(i_{\mathrm{s}}^{d},\, i_{\mathrm{s}}^{q}\right) = \tfrac{3}{2}\, n_{\mathrm{p}}\, \psi_{\mathrm{pm}} i_{\mathrm{s}}^{q} \quad \left(\Longrightarrow \; i_{\mathrm{s,ref}}^{q} = \frac{2\, m_{\mathrm{m,ref}}}{3 n_{\mathrm{p}} \psi_{\mathrm{pm}}}\right)$$

Motivation

Optimal feedforward torque control problem: **Anisotropic machine** (with cross-coupling)

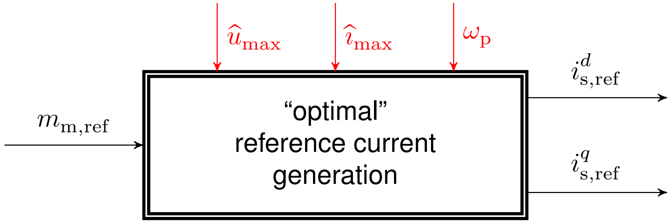


$$\color{red}{m_\mathrm{m,ref}} \overset{!}{=} m_\mathrm{m}\big(i_\mathrm{s}^d,\, i_\mathrm{s}^q\big) = \tfrac{3}{2}\, n_\mathrm{p} \Big[\underbrace{\psi_\mathrm{pm} i_\mathrm{s}^q}_{\text{magnetic torque}} + \underbrace{\big(L_\mathrm{s}^d - L_\mathrm{s}^q\big) i_\mathrm{s}^d i_\mathrm{s}^q}_{\text{reluctance torque}} + \underbrace{L_\mathrm{s,m}\big((i_\mathrm{s}^q)^2 - (i_\mathrm{s}^d)^2\big)}_{\text{cross-coupling torque}} \Big]$$

Outline

Problem statement and proposed solution

Optimal feedforward torque control problem: Optimal reference current generation

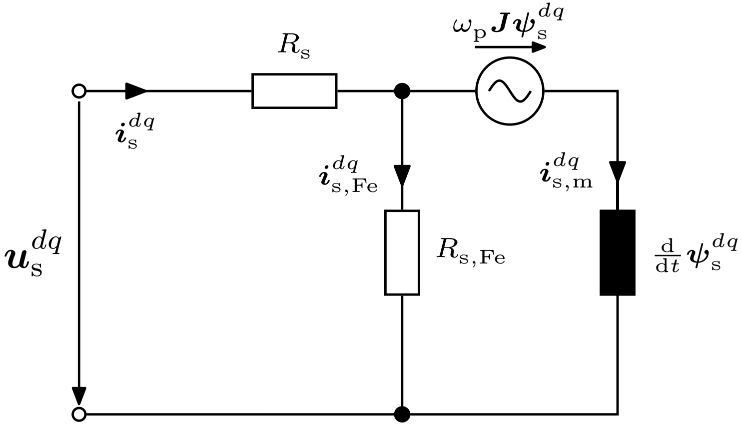


$$\boldsymbol{i}^{dq}_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_{\mathrm{p}}) = \begin{pmatrix} i^{d}_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_{\mathrm{p}}) \\ i^{q}_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_{\mathrm{p}}) \end{pmatrix}$$

- ✓ Numerical solutions and/or look-up tables (but: limited storage, accuracy, real-time applicability)
- ? Analytical solutions (some do exist but impose simplifying assumptions such as ~~$R_{\mathrm{s}} = 0$, $R_{\mathrm{Fe}} = 0$ and/or $L_{\mathrm{s,m}} = 0$~~)

Problem statement and proposed solution

Considered anisotropic synchronous machines with iron losses [2]: Electrical circuit



Problem statement and proposed solution

Considered anisotropic synchronous machines with iron losses [2]: Modeling

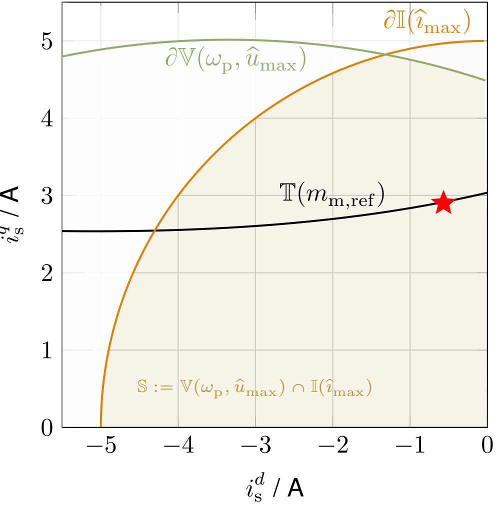
$$
\begin{aligned}
\overbrace{\begin{pmatrix} u_{\mathrm{s}}^{d} \\ u_{\mathrm{s}}^{q} \end{pmatrix}}^{=:\boldsymbol{u}_{\mathrm{s}}^{dq}} &= R_{\mathrm{s}}(\vartheta)\overbrace{\begin{pmatrix} i_{\mathrm{s}}^{d} \\ i_{\mathrm{s}}^{q} \end{pmatrix}}^{=:\boldsymbol{i}_{\mathrm{s}}^{dq}} + \omega_{\mathrm{p}} \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{=:\boldsymbol{J}} \overbrace{\begin{pmatrix} \psi_{\mathrm{s}}^{d}\left(\boldsymbol{i}_{\mathrm{s,m}}^{dq}\right) \\ \psi_{\mathrm{s}}^{q}\left(\boldsymbol{i}_{\mathrm{s,m}}^{dq}\right) \end{pmatrix}}^{=:\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq})} + \cancel{\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq})} \\
\boldsymbol{i}_{\mathrm{s}}^{dq} &= \boldsymbol{i}_{\mathrm{s,m}}^{dq} + \boldsymbol{i}_{\mathrm{s,Fe}}^{dq} = \boldsymbol{i}_{\mathrm{s,m}}^{dq} + \frac{\omega_{\mathrm{p}}\boldsymbol{J}\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq}) + \cancel{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq})}}{R_{\mathrm{Fe}}(\omega_{\mathrm{p}})} \\
p_{\mathrm{loss}} &= p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) + p_{\mathrm{Fe}}(\boldsymbol{\psi}_{\mathrm{s}}^{dq}, \omega_{\mathrm{p}}) = \tfrac{3}{2}R_{\mathrm{s}}\|\boldsymbol{i}_{\mathrm{s}}^{dq}\|^{2} + \frac{3}{2}\frac{\|\omega_{\mathrm{p}}\boldsymbol{J}\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq}) + \cancel{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s,m}}^{dq})}}{R_{\mathrm{Fe}}(\omega_{\mathrm{p}})} \\
m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s,m}}^{dq}) &= \tfrac{3}{2}n_{\mathrm{p}}\left(\boldsymbol{i}_{\mathrm{s,m}}^{dq}\right)^{\top}\boldsymbol{J}\boldsymbol{\psi}_{\mathrm{s}}^{dq}\left(\boldsymbol{i}_{\mathrm{s,m}}^{dq}\right)
\end{aligned}
$$

Assumption (Affine flux linkage (at least locally))

$$\boldsymbol{\psi}_{\mathrm{s}}^{dq}\left(\boldsymbol{i}_{\mathrm{s,m}}^{dq}\right) \approx \underbrace{\begin{bmatrix} L_{\mathrm{s}}^{d} & L_{\mathrm{s,m}} \\ L_{\mathrm{s,m}} & L_{\mathrm{s}}^{q} \end{bmatrix}}_{=:\boldsymbol{L}_{\mathrm{s}}^{dq} \in \mathbb{R}^{2\times 2}} \boldsymbol{i}_{\mathrm{s,m}}^{dq} + \underbrace{\begin{pmatrix} \psi_{\mathrm{pm}} \\ 0 \end{pmatrix}}_{=:\boldsymbol{\psi}_{\mathrm{pm}}^{dq}} \quad \text{with } \boldsymbol{L}_{\mathrm{s}}^{dq} = \left(\boldsymbol{L}_{\mathrm{s}}^{dq}\right)^{\top} > 0$$

Problem statement and proposed solution

Optimal reference current generation: Optimization problem(s) with multiple constraints



$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} \; -f(\boldsymbol{i}_{\mathrm{s}}^{dq}) \quad \text{subject to}$$

$$\begin{cases} \|\boldsymbol{i}_{\mathrm{s}}^{dq}\|^2 \leqslant \widehat{\imath}_{\max}^2, \\ \text{(current circular area)} \\ \|\boldsymbol{u}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s}}^{dq}, \omega_{\mathrm{p}})\|^2 \leqslant \widehat{u}_{\max}^2, \\ \text{(voltage elliptical area)} \\ |m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})| \leqslant |m_{\mathrm{m,ref}}|, \text{ and} \\ \mathrm{sign}(m_{\mathrm{m,ref}}) = \\ \mathrm{sign}(m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})). \end{cases}$$

e.g. $-f(\boldsymbol{i}_{\mathrm{s}}^{dq}) = -\|\boldsymbol{i}_{\mathrm{s}}^{dq}\|^2$
(minimize copper losses)
$\Longrightarrow \; \boldsymbol{i}_{\mathrm{s,ref}}^{dq}$ at ★

Problem statement and proposed solution

Proposed solution: Implicit formulation (quadrics), optimization and intersection points

Step 1: Derivation of quadrics $Q_A(\boldsymbol{i}_s^{dq}) := (\boldsymbol{i}_s^{dq})^\top \boldsymbol{A}\, \boldsymbol{i}_s^{dq} + 2\boldsymbol{a}^\top \boldsymbol{i}_s^{dq} + \alpha$:

- Current circular area: $(i_s^d)^2 + (i_s^q)^2 \leqslant \widehat{\imath}_{\max}^2 \iff \|\boldsymbol{i}_s^{dq}\|^2 = (\boldsymbol{i}_s^{dq})^\top \boldsymbol{I}_2 \boldsymbol{i}_s^{dq} \leqslant \widehat{\imath}_{\max}^2$

$$\Longrightarrow \quad \mathbb{I}(\widehat{\imath}_{\max}) := \left\{ \boldsymbol{i}_s^{dq} \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_s^{dq})^\top \boldsymbol{I}_2 \boldsymbol{i}_s^{dq} - \widehat{\imath}_{\max}^2 \leqslant 0 \right\}$$

- Voltage elliptical area: $(u_s^d)^2 + (u_s^q)^2 \leqslant \widehat{u}_{\max}^2 \iff \|\boldsymbol{u}_s^{dq}(\boldsymbol{i}_s^{dq}, \omega_p)\|^2 = \ldots \leqslant \widehat{u}_{\max}^2$

$$\Longrightarrow \quad \mathbb{V}(\omega_p, \widehat{u}_{\max}) := \left\{ \boldsymbol{i}_s^{dq} \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_s^{dq})^\top \boldsymbol{V}(\omega_p)\, \boldsymbol{i}_s^{dq} + 2\,\boldsymbol{v}(\omega_p)^\top \boldsymbol{i}_s^{dq} + \nu(\omega_p, \widehat{u}_{\max}) \leqslant 0 \right.$$

- Reference torque hyperbola:

$m_m(\boldsymbol{i}_{s,m}^{dq}) = \frac{3}{2} n_p (\boldsymbol{i}_{s,m}^{dq})^\top \boldsymbol{J} \boldsymbol{\psi}_s^{dq}(\boldsymbol{i}_{s,m}^{dq}) \overset{!}{=} m_{m,\mathrm{ref}} \iff \ldots$

$$\Longrightarrow \quad \mathbb{T}(m_{m,\mathrm{ref}}) := \left\{ \boldsymbol{i}_s^{dq} \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_s^{dq})^\top \boldsymbol{T} \boldsymbol{i}_s^{dq} + 2\,\boldsymbol{t}^\top \boldsymbol{i}_s^{dq} - m_{m,\mathrm{ref}} = 0 \right\}$$

- ...

Problem statement and proposed solution

Proposed solution: Implicit formulation (quadrics), optimization and intersection points

Step 2: Optimization problem with equality constraint

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq}} -\Big(\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{A} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\boldsymbol{a}^\top \boldsymbol{i}_{\mathrm{s}}^{dq} + \alpha}_{=:Q_A(\boldsymbol{i}_{\mathrm{s}}^{dq})} \Big) \text{ s.t. } \underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{B} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\boldsymbol{b}^\top \boldsymbol{i}_{\mathrm{s}}^{dq} + \beta}_{=:Q_B(\boldsymbol{i}_{\mathrm{s}}^{dq})} = 0$$

$\Longrightarrow$ Hyperbolas for e.g. MTPC, MTPL or MTPV (with $R_{\mathrm{s}}, R_{\mathrm{Fe}} \neq 0$ & $L_{\mathrm{s,m}} \neq 0$)

Step 3: Intersection of two quadrics (e.g. voltage ellipse and current circle)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} := \arg\min_{\|\boldsymbol{i}_{\mathrm{s}}^{dq}\|} \left\{ \boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{R}^2 \;\middle|\; Q_A(\boldsymbol{i}_{\mathrm{s}}^{dq}) = 0 \;\wedge\; Q_B(\boldsymbol{i}_{\mathrm{s}}^{dq}) = 0 \right\}$$

$\Longrightarrow$ Optimal operation point ★ (reference current)

Both lead to subproblem of solving a fourth-order (quartic) polynomial

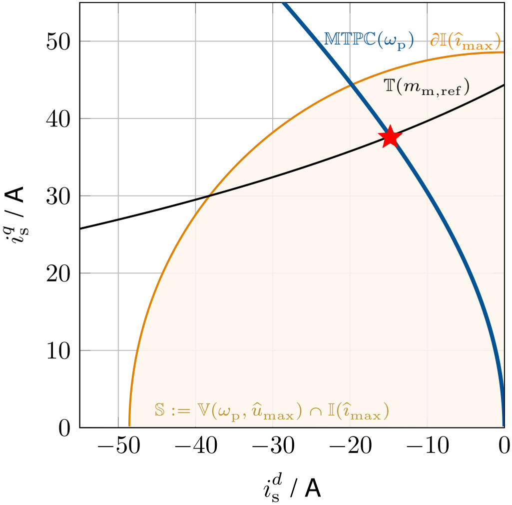
$$\chi(\lambda) := c_4\,\lambda^4 + c_3\,\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 \stackrel{!}{=} 0$$

$\Longrightarrow$ Analytical solutions exist (e.g. Euler's solution, see [3])

Outline

Optimal operation strategies

Maximum Torque per Current (**MTPC @** $0 \cdot \omega_{\mathrm{m,R}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^{\top} \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\, \boldsymbol{t}^{\top} \boldsymbol{i}_{\mathrm{s}}^{dq}}_{= m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})} - m_{\mathrm{m,ref}} = 0$$

Solution set

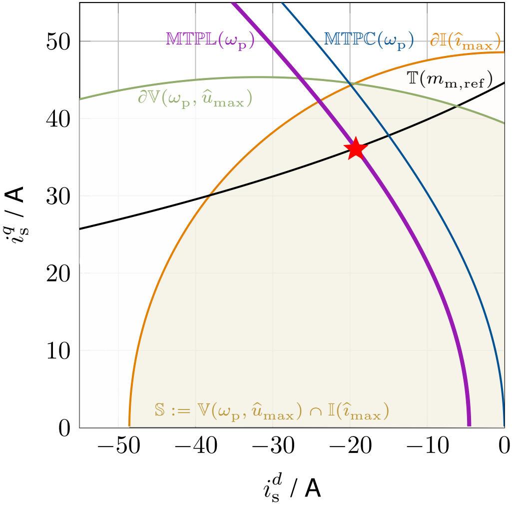
$$\mathbb{MTPC}(\omega_{\mathrm{p}}) := \left\{ \boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{R}^2 \,\middle|\, (\boldsymbol{i}_{\mathrm{s}}^{dq})^{\top} \boldsymbol{M}_{\mathrm{C}}\, \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\, \boldsymbol{m}_{\mathrm{C}}^{\top} \boldsymbol{i}_{\mathrm{s}}^{dq} = 0 \right\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \mathbb{MTPC} \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Maximum Torque per Losses (**MTPL** @ $1 \cdot \omega_{\mathrm{m,R}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) - p_{\mathrm{Fe}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^{\top}\boldsymbol{T}\boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{t}^{\top}\boldsymbol{i}_{\mathrm{s}}^{dq}}_{=m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})} - m_{\mathrm{m,ref}} = 0$$

Solution set

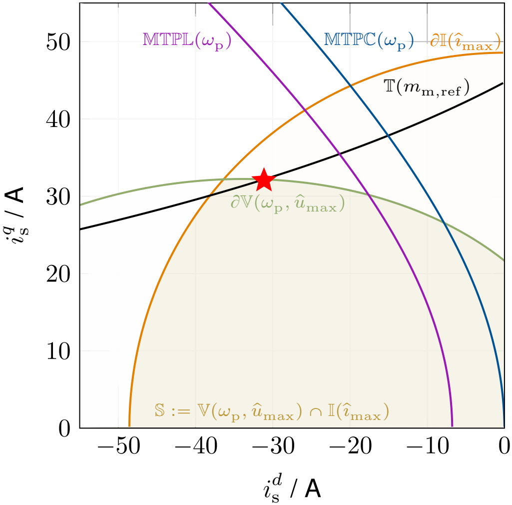
$$\mathrm{MTPL}(\omega_{\mathrm{p}}) := \Big\{\, \boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{R}^2 \;\Big|\; (\boldsymbol{i}_{\mathrm{s}}^{dq})^{\top}\boldsymbol{M}_{\mathrm{L}}\,\boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{m}_{\mathrm{L}}^{\top}\boldsymbol{i}_{\mathrm{s}}^{dq} + \mu_{\mathrm{L}} = 0 \Big\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \mathrm{MTPL} \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Field Weakening (**FW** @ $1.40 \cdot \omega_{\mathrm{m,R}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) - p_{\mathrm{Fe}}(\boldsymbol{i}_{\mathrm{s}}^{dq}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^{\top} \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{t}^{\top} \boldsymbol{i}_{\mathrm{s}}^{dq}}_{= m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})} - m_{\mathrm{m,ref}} = 0$$

Feasible set

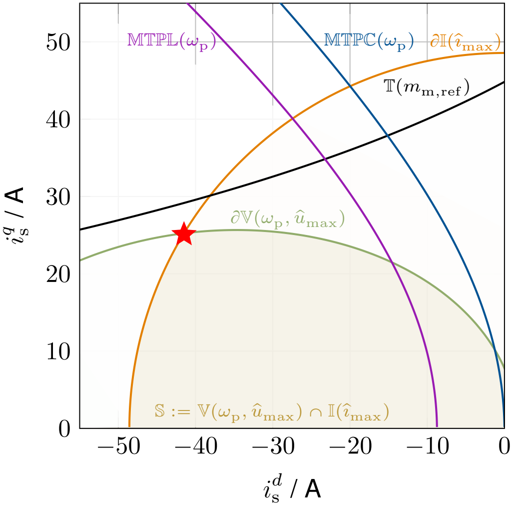
$$\mathbb{FW}(m_{\mathrm{m,ref}}, \omega_{\mathrm{p}}, \widehat{u}_{\max}) := \mathbb{S}(\omega_{\mathrm{p}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}) \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \partial\mathbb{V}(\omega_{\mathrm{p}}, \widehat{u}_{\max}) \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Maximum Current (**MC** @ $1.75 \cdot \omega_{\mathrm{m,R}}$): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})$$

Feasible set

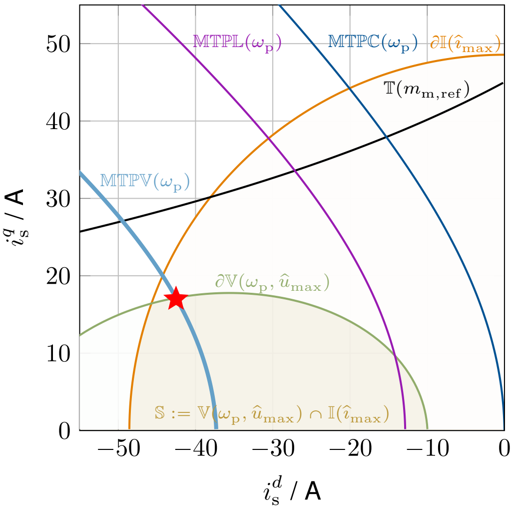
$$\mathbb{MC}(\omega_{\mathrm{p}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}) := \mathbb{V}(\omega_{\mathrm{p}}, \widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Optimal current reference (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \partial\mathbb{V}(\omega_{\mathrm{p}}, \widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Optimal operation strategies

Maximum Torque per Voltage (**MTPV @** $2.50 \cdot \omega_{\mathrm{m,R}}$): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} -\|\boldsymbol{u}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s}}^{dq})\|^2 \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{t}^\top \boldsymbol{i}_{\mathrm{s}}^{dq}}_{=m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})} - m_{\mathrm{m,ref}} = 0$$

Solution set

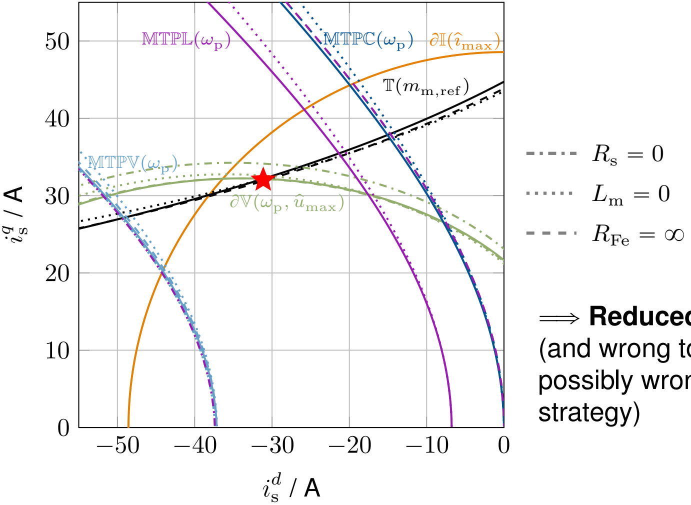
$$\mathbb{MTPV}(\omega_{\mathrm{p}}) := \left\{ \boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{M}_{\mathrm{V}}\, \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{m}_{\mathrm{V}}^\top \boldsymbol{i}_{\mathrm{s}}^{dq} + \mu_{\mathrm{V}} = 0 \right\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \mathbb{MTPV}(\omega_{\mathrm{p}}) \cap \partial\mathbb{V}(\omega_{\mathrm{p}}, \hat{u}_{\max})$$

Optimal operation strategies

Field Weakening (**FW** @ $1.40 \cdot \omega_{\mathrm{m,R}}$): Illustration of the effect of neglecting R_{s}, R_{Fe} and/or L_{m}



$\Longrightarrow$ **Reduced efficiency**! (and wrong torque and/or possibly wrong operation strategy)

Outline

4 Analytical solution

Analytical solution

Example: Analytical solution for intersection points of two quadrics

Problem

$$\mathbb{X} := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \underbrace{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^\top \boldsymbol{x} + \alpha}_{=:Q_A(\boldsymbol{x})} = 0 \;\wedge\; \underbrace{\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + \beta}_{=:Q_B(\boldsymbol{x})} = 0 \right\}$$

Solution: Relate both (e.g., if $\alpha \neq 0$ and $\beta \neq 0$)

$$Q_D(\boldsymbol{x}) := \frac{Q_A(\boldsymbol{x})}{\alpha} - \frac{Q_B(\boldsymbol{x})}{\beta} = \boldsymbol{x}^\top \underbrace{\left(\frac{\boldsymbol{A}}{\alpha} - \frac{\boldsymbol{B}}{\beta}\right)}_{=:\boldsymbol{D}} \boldsymbol{x} + 2 \underbrace{\left(\frac{\boldsymbol{a}}{\alpha} - \frac{\boldsymbol{b}}{\beta}\right)^\top}_{=:\boldsymbol{d}^\top} \boldsymbol{x} + \underbrace{\left(\frac{\alpha}{\alpha} - \frac{\beta}{\beta}\right)}_{=0} = 0$$

and note that

$$Q_D(\boldsymbol{x}) := \boldsymbol{x}^\top \big(\underbrace{\boldsymbol{D}\boldsymbol{x} + 2\boldsymbol{d}}_{\stackrel{!}{=}\lambda \boldsymbol{J}\boldsymbol{x}}\big) \stackrel{!}{=} 0 \quad \Longleftrightarrow \quad \boldsymbol{x}^\star(\lambda) = -2\big(\boldsymbol{D} - \lambda \boldsymbol{J}\big)^{-1} \boldsymbol{d}$$

Analytical solution

Example: Analytical solution for intersection points of two quadrics (cont'd)

then insert $\boldsymbol{x}^\star(\lambda)$ in $Q_A(\boldsymbol{x})$ (or $Q_B(\boldsymbol{x})$), i.e.

$$Q_A(\boldsymbol{x}^\star(\lambda)) =$$

$$4\boldsymbol{d}^\top\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-\top}\boldsymbol{A}\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-1}\boldsymbol{d} - 4\boldsymbol{a}^\top\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-1}\boldsymbol{d} + \alpha = 0 \,\Big|\, \cdot \big[\underbrace{\det\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)}_{\propto\lambda^2}\big]^2$$

$$\Longrightarrow \boxed{\chi(\lambda) := c_4\,\lambda^4 + c_3\,\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 \overset{!}{=} 0}$$

$\Longrightarrow$ Analytical (real) solutions $\lambda^\star \in \{\lambda_1, \ldots, \lambda_4\}$ can be computed [3]:

$$\Longrightarrow \boxed{\boldsymbol{x}^\star(\lambda^\star) = -2\big(\boldsymbol{D} - \lambda^\star\boldsymbol{J}\big)^{-1}\,\boldsymbol{d}}$$

Outline

Measurement results

Laboratory setup

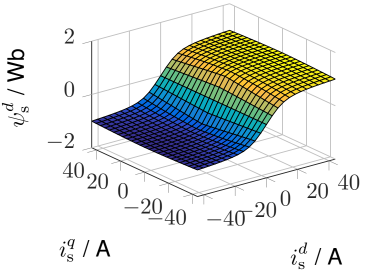
Reluctance SM (RSM) and Permanent-magnet SM

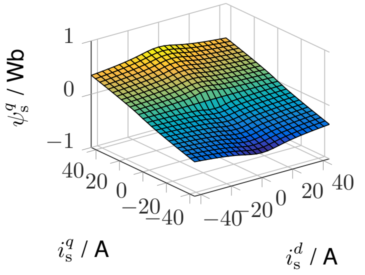
Host PC (rapid prototyping)

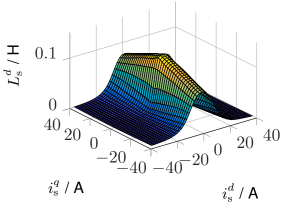
Real-time system and VSIs

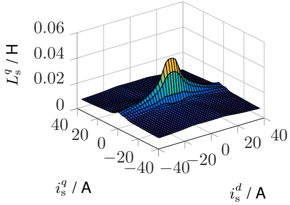
Measurement results

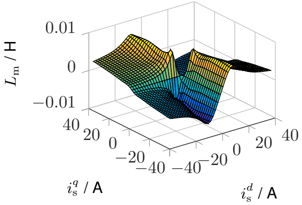
RSM: Nonlinear flux linkages and differential inductances





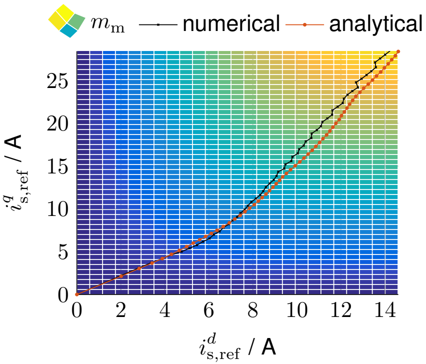




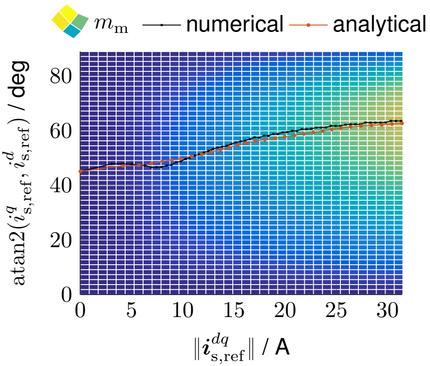


Measurement results

RSM: Comparison of numerical and analytical solutions for MTPC



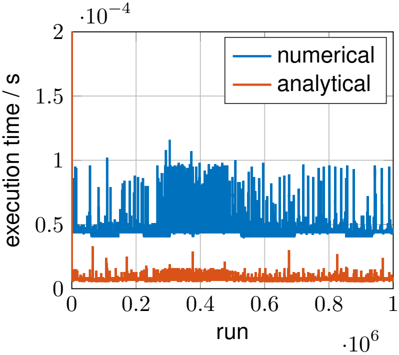
Cartesian coordinates

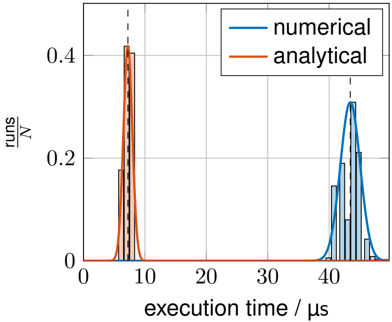


polar coordinates

Measurement results

RSM: Comparison of computational load

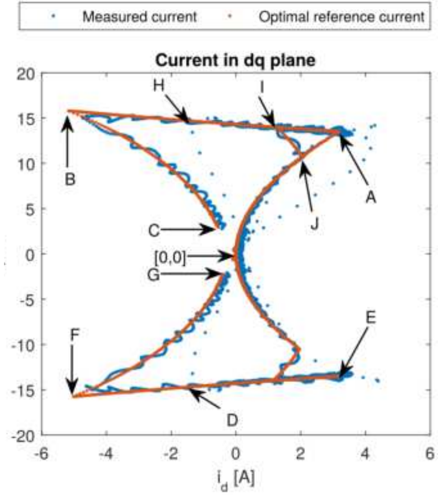


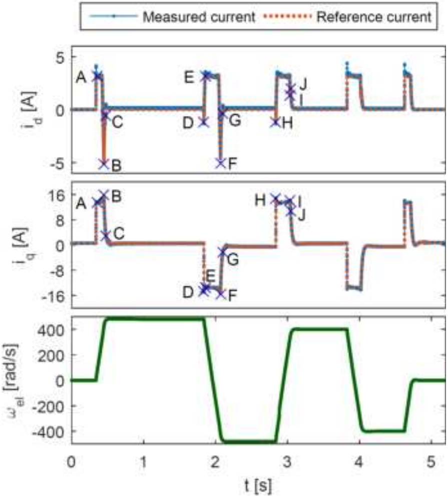


- Execution times for $N = 10^6$ runs (downsampled by factor 100)
- Standard deviations $\sigma_n = 1{,}61 \cdot 10^{-6}$ s and $\sigma_a = 0{,}73 \cdot 10^{-6}$ s
- Average execution times $\mu_n = 43{,}4 \cdot 10^{-6}$ s and $\mu_a = 7{,}23 \cdot 10^{-6}$ s ($>$**6x faster**)

Measurement results

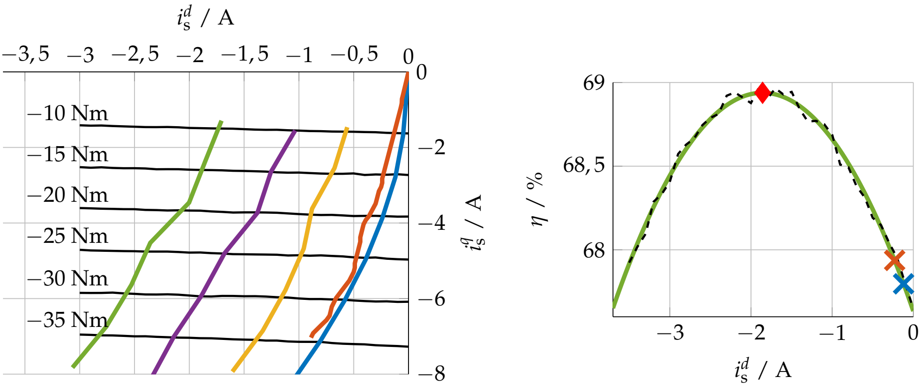
PME-RSM: Implementation by colleagues ($p_{\mathrm{m,R}} = 4{,}5\,\mathrm{kW}$ & $L_s^d/L_s^q = 1.43$) [4]





Measurement results

IPMSM (generator mode): Efficiency enhancements considering iron losses [5]



Outline

Conclusion

Summary and future work

To take home

- **unified** theory for MTPC, MTPL, FW, MC and MTPV based on [6]
  - **quadrics** (implicit formulation),
  - **optimization problems** with **equality constraint**, and
  - **intersection** of two quadrics
- **analytical** solution for reference currents (& transition points)
- $R_{\mathrm{s}} \neq 0$, $R_{\mathrm{Fe}} \neq 0$ and $L_{\mathrm{m}} \neq 0$ (can all be considered **simultaneously**)
- **fast(er)** and **more accurate** computation
- **applicable** in real-world (e.g. nonlinear RSM, PME-RSM or IPMSM; also IMs)

Future work

- convergence (mathematical proof for nonlinear flux linkages)
- more extensive experimental validation
- impact of (parameter) uncertainties
- consideration of *rotor* iron losses (e.g. for induction machines)
- combination with minimization of *conduction and switching losses*

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Journal article (published 2017)

INTERNATIONAL JOURNAL OF CONTROL, 2017
https://doi.org/10.1080/00207179.2017.1338359

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A unified theory for optimal feedforward torque control of anisotropic synchronous machines

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ABSTRACT

A unified theory for optimal feedforward torque control of anisotropic synchronous machines with *non-negligible stator resistance* and *mutual inductance* is presented which allows to *analytically* compute (1) the optimal direct and quadrature reference currents for all operating strategies, such as maximum torque per current (MTPC), maximum current, field weakening, maximum torque per voltage (MTPV) or maximum torque per flux (MTPF), and (2) the transition points indicating when to switch between the operating strategies due to speed, voltage or current constraints. The analytical solutions allow for an (almost) instantaneous selection and computation of actual operation strategy and corresponding reference currents. Numerical methods (approximating these solutions only) are *no longer* required. The unified theory is based on one simple idea: all optimisation problems, their respective constraints and the computation of the intersection point(s) of voltage ellipse, current circle or torque, MTPC, MTPV, MTPF hyperbolas are reformulated implicitly as *quadrics* which allows to invoke the Lagrangian formalism and to find the roots of fourth-order polynomials analytically. The proposed theory is suitable for any anisotropic synchronous machine. Implementation and measurement results illustrate effectiveness and applicability of the theoretical findings in real world.

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KEYWORDS
Maximum torque per ampere (MTPA); maximum torque per current (MTPC); maximum torque per voltage (MTPV); maximum torque per flux (MTPF); maximum current (MC); field weakening (FW); analytical solution; optimal feedforward torque control; efficiency; copper losses; anisotropy; synchronous machine; interior

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Monograph (published May 2017)

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Book chapter (to be published in 2020 in Schröder “Elektrische Antriebssysteme”)

6.9 Optimale Betriebsführung von nichtlinearen Synchronmaschinen

C. M. Hackl, J. Kullick, N. Monzen

Der folgende Abschnitt basiert auf den Publikationen [HH16; Eld+16; EHK16; Eld+17b; Hac+17; Eld+17a]. Darin wurde die erste allgemeine Theorie zur analytischen Berechnung der optimalen (verlustminimierenden) Sollströme für anisotrope Synchronmaschinen mit konstanter Erregung vorgestellt. Die weithin verbreiteten und vereinfachenden Annahmen wie z. B. die Vernachlässigung des Statorwiderstandes oder der magnetischen Kreuzkopplung konnten aufgehoben werden. Zusätzlich erlaubt die Theorie die Berücksichtigung von Eisenverlusten und Nichtlinearitäten in der Maschine, wie die Erweiterung[11] in diesem Beitrag zeigen wird. Es werden im Folgenden Motivation, Modellierung, Problemstellung, Idee der vorgeschlagenen Lösung, mathematische Grundlagen, Implementierungsaspekte und

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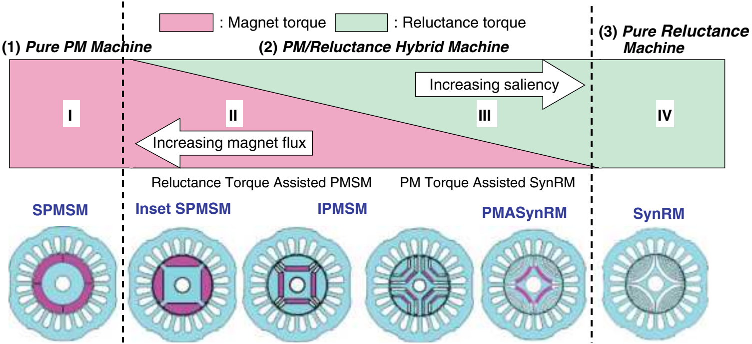
Outline

Motivation

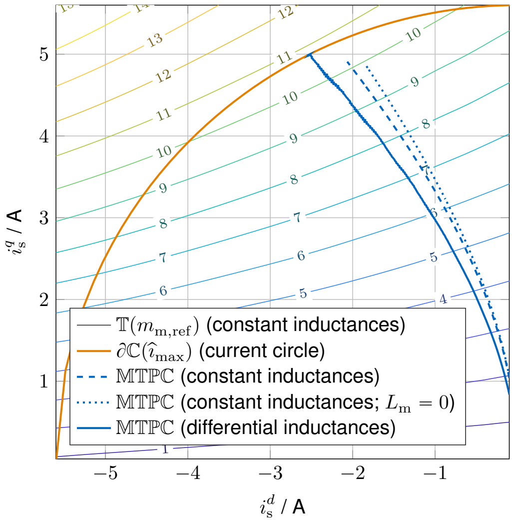
Examples of anisotropic synchronous machines [1] with “saliency ratio” $L_s^d / L_s^q \neq 1$



$$m_m(i_s^d, i_s^q) = \frac{3}{2} n_p \Big[\underbrace{\psi_{pm} i_s^q}_{\text{magnetic torque}} + \underbrace{(L_s^d - L_s^q) i_s^d i_s^q}_{\text{reluctance torque}} + \underbrace{L_{s,m}\big((i_s^q)^2 - (i_s^d)^2\big)}_{\text{cross-coupling torque}} \Big]$$

Implementation results

IPMSM: MTPC with non-negligible mutual inductance and nonlinear flux linkage



Analytical solution

Example: Analytical solution for intersection points of two quadrics

Problem

$$\mathbb{X} := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \underbrace{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^\top \boldsymbol{x} + \alpha}_{=: Q_A(\boldsymbol{x})} = 0 \;\wedge\; \underbrace{\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + \beta}_{=: Q_B(\boldsymbol{x})} = 0 \right\}$$

Solution: Relate both (e.g., if $\alpha \neq 0$ and $\beta \neq 0$)

$$Q_D(\boldsymbol{x}) := \frac{Q_A(\boldsymbol{x})}{\alpha} - \frac{Q_B(\boldsymbol{x})}{\beta} = \boldsymbol{x}^\top \underbrace{\left(\frac{\boldsymbol{A}}{\alpha} - \frac{\boldsymbol{B}}{\beta}\right)}_{=: \boldsymbol{D}} \boldsymbol{x} + 2 \underbrace{\left(\frac{\boldsymbol{a}}{\alpha} - \frac{\boldsymbol{b}}{\beta}\right)^\top}_{=: \boldsymbol{d}^\top} \boldsymbol{x} + \underbrace{\left(\frac{\alpha}{\alpha} - \frac{\beta}{\beta}\right)}_{=0} = 0$$

and note that

$$Q_D(\boldsymbol{x}) := \boldsymbol{x}^\top \big(\underbrace{\boldsymbol{D}\boldsymbol{x} + 2\boldsymbol{d}}_{\stackrel{!}{=} \lambda \boldsymbol{J}\boldsymbol{x}} \big) \stackrel{!}{=} 0 \quad \Longleftrightarrow \quad \boldsymbol{x}^\star(\lambda) = -2\big(\boldsymbol{D} - \lambda \boldsymbol{J}\big)^{-1} \boldsymbol{d}$$

Analytical solution

Example: Analytical solution for intersection points of two quadrics (cont'd)

then insert $\boldsymbol{x}^\star(\lambda)$ in $Q_A(\boldsymbol{x})$ (or $Q_B(\boldsymbol{x})$), i.e.

$$Q_A(\boldsymbol{x}^\star(\lambda)) =$$

$$4\boldsymbol{d}^\top\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-\top}\boldsymbol{A}\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-1}\boldsymbol{d} - 4\boldsymbol{a}^\top\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)^{-1}\boldsymbol{d} + \alpha = 0 \,\Big|\, \cdot \Big[\underbrace{\det\big(\boldsymbol{D}-\lambda\boldsymbol{J}\big)}_{\propto\lambda^2}\Big]^2$$

$$\Longrightarrow \boxed{\chi(\lambda) := c_4\,\lambda^4 + c_3\,\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 \overset{!}{=} 0}$$

$\Longrightarrow$ Analytical (real) solutions $\lambda^\star \in \{\lambda_1, \ldots, \lambda_4\}$ can be computed [3]:

$$\Longrightarrow \boxed{\boldsymbol{x}^\star(\lambda^\star) = -2\big(\boldsymbol{D} - \lambda^\star\boldsymbol{J}\big)^{-1}\,\boldsymbol{d}}$$

Analytical solution

Example: Analytical solution for optimization problem with quadrics

Problem

$$\max_{\boldsymbol{i}_\mathrm{s}^{dq}} -\Big(\underbrace{(\boldsymbol{i}_\mathrm{s}^{dq})^\top \boldsymbol{A}\boldsymbol{i}_\mathrm{s}^{dq} + 2\boldsymbol{a}^\top \boldsymbol{i}_\mathrm{s}^{dq} + \alpha}_{=:Q_A(\boldsymbol{i}_\mathrm{s}^{dq})}\Big) \text{ s.t. } \underbrace{(\boldsymbol{i}_\mathrm{s}^{dq})^\top \boldsymbol{B}\boldsymbol{i}_\mathrm{s}^{dq} + 2\boldsymbol{b}^\top \boldsymbol{i}_\mathrm{s}^{dq} + \beta}_{=:Q_B(\boldsymbol{i}_\mathrm{s}^{dq})} = 0$$

$\Longrightarrow$ Hyperbolas for MTPC, MTPV and MPTF (with $R_\mathrm{s} \neq 0$ & $L_{\mathrm{s,m}} \neq 0$!)

Lagrangian with Lagrangian multiplier $\lambda \in \mathbb{C}$

$$\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq}, \lambda) := -Q_A(\boldsymbol{i}_\mathrm{s}^{dq}) + \lambda\, Q_B(\boldsymbol{i}_\mathrm{s}^{dq})$$

Gradient (necessary conditions)

$$\boldsymbol{g}_\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq}, \lambda) := \begin{pmatrix} \left(\dfrac{\mathrm{d}^{\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq},\lambda)}}{\mathrm{d}\boldsymbol{i}_\mathrm{s}^{dq\,\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq},\lambda)}}\right)^\top \\ \dfrac{\mathrm{d}^{\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq},\lambda)}}{\mathrm{d}\lambda^{\mathfrak{L}(\boldsymbol{i}_\mathrm{s}^{dq},\lambda)}} \end{pmatrix} \stackrel{!}{=} \boldsymbol{0}_3 \Leftrightarrow \begin{pmatrix} -2\boldsymbol{A}\boldsymbol{i}_\mathrm{s}^{dq} - 2\boldsymbol{a} + \lambda(2\boldsymbol{B}\boldsymbol{i}_\mathrm{s}^{dq} + 2\boldsymbol{b}) \\ (\boldsymbol{i}_\mathrm{s}^{dq})^\top \boldsymbol{B}\boldsymbol{i}_\mathrm{s}^{dq} + 2\boldsymbol{b}^\top \boldsymbol{i}_\mathrm{s}^{dq} + \beta \end{pmatrix} \stackrel{!}{=} \boldsymbol{0}_3.$$

Analytical solution

Example: Analytical solution for optimization problem with quadrics (cont'd)

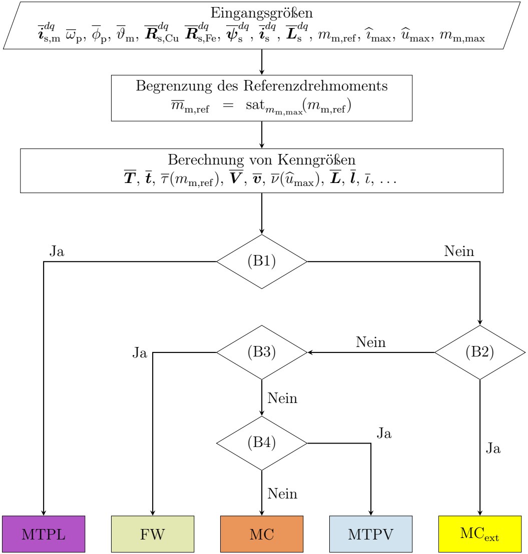
Hessian (sufficient conditions)

$$\boldsymbol{H}_{\mathfrak{L}}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda) := \frac{\mathrm{d}}{\mathrm{d}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda)}\left(\frac{\mathrm{d}\,\mathfrak{L}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda)}{\mathrm{d}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda)}\mathfrak{L}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda)\right)^{\top}$$

$$= \begin{bmatrix} 2\boldsymbol{M}(\lambda) & 2\boldsymbol{B}\boldsymbol{i}_{\mathrm{s}}^{dq} + 2\boldsymbol{b} \\ (2\boldsymbol{B}\boldsymbol{i}_{\mathrm{s}}^{dq} + 2\boldsymbol{b})^{\top} & 0 \end{bmatrix} = \boldsymbol{H}_{\mathfrak{L}}(\boldsymbol{i}_{\mathrm{s}}^{dq},\lambda)^{\top} \overset{!}{<} 0$$

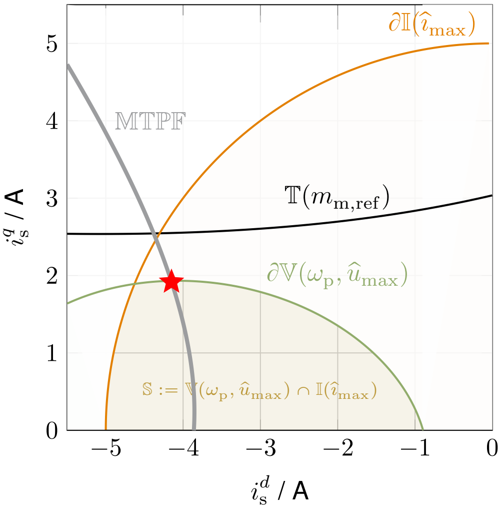
Operation management and operation strategies

Decision tree (transition between operation strategies)



Operation management and operation strategies

Maximum Torque per Flux (**MTPF**): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} -\|\boldsymbol{\psi}_{\mathrm{s}}^{dq}(\boldsymbol{i}_{\mathrm{s}}^{dq})\|^2 \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{t}^\top \boldsymbol{i}_{\mathrm{s}}^{dq}}_{=m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})} - m_{\mathrm{m,ref}} = 0$$

Solution set

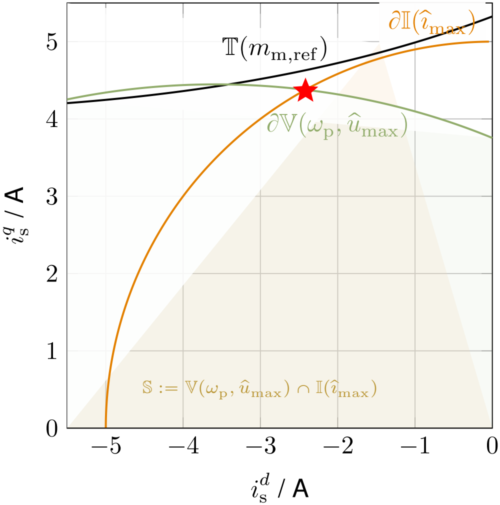
$$\mathrm{MTPF} := \Big\{ \boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{R}^2 \;\Big|\; (\boldsymbol{i}_{\mathrm{s}}^{dq})^\top \boldsymbol{M}_{\mathrm{F}}\, \boldsymbol{i}_{\mathrm{s}}^{dq} + 2\,\boldsymbol{m}_{\mathrm{F}}^\top \boldsymbol{i}_{\mathrm{s}}^{dq} + \mu_{\mathrm{F}} = 0 \Big\}$$

Optimal current reference (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \mathrm{MTPF} \cap \partial\mathbb{V}(\omega_{\mathrm{p}}, \widehat{u}_{\max})$$

Operation management and operation strategies

Maximum Current (**MC**): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{dq} \in \mathbb{S}} m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{dq})$$

Feasible set

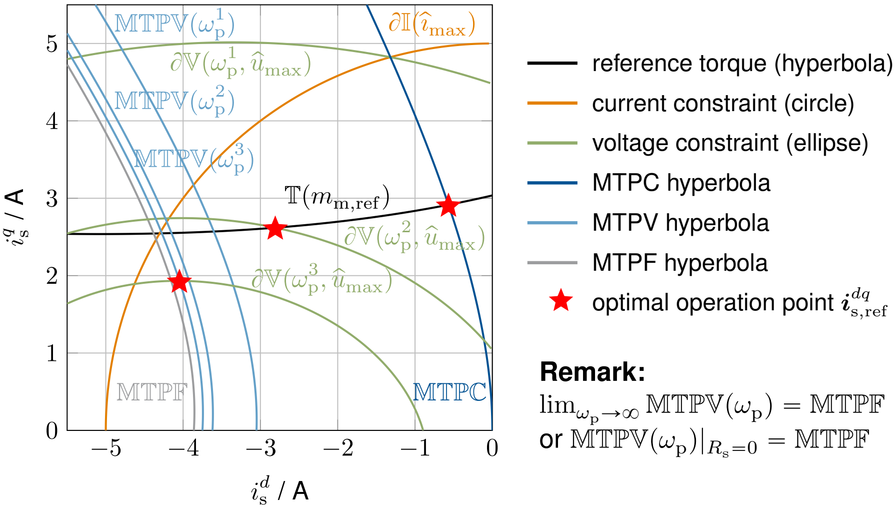
$$\mathbb{MC}(\omega_{\mathrm{p}},\widehat{u}_{\max},\widehat{\imath}_{\max}) := \mathbb{V}(\omega_{\mathrm{p}},\widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Optimal current reference (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{dq} = \partial\mathbb{V}(\omega_{\mathrm{p}},\widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Operation management and operation strategies

Illustration of MTPC ($\omega_\mathrm{p}^1 = \omega_\mathrm{p,R}$), FW ($\omega_\mathrm{p}^2 = 2\omega_\mathrm{p,R}$) and MTPV ($\omega_\mathrm{p}^3 = 3\omega_\mathrm{p,R}$)



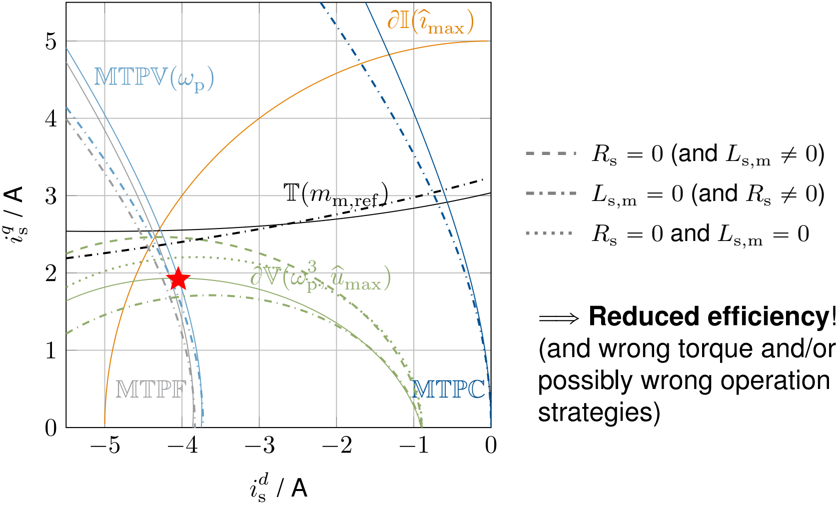
Remark:

$\lim_{\omega_\mathrm{p}\to\infty} \mathrm{MTPV}(\omega_\mathrm{p}) = \mathrm{MTPF}$

or $\mathrm{MTPV}(\omega_\mathrm{p})|_{R_\mathrm{s}=0} = \mathrm{MTPF}$

Operation management and operation strategies

Illustration of the effect of neglecting R_s and/or $L_{\mathrm{s,m}}$



$\Longrightarrow$ **Reduced efficiency**!
(and wrong torque and/or possibly wrong operation strategies)

Roots of a fourth-order polynomial

Euler's solution [3]

Problem: Find roots of

$$\chi(\lambda) := c_4\,\lambda^4 + c_3\,\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 \overset{!}{=} 0$$

Solution:
Step 1: Transform $\chi(\lambda)$ to *depressed* polynomial $\chi_{\rm dep}(y)$

$$\chi_{\rm dep}(y) := y^4 + p\,y^2 + q\,y + r \overset{!}{=} 0 \quad \text{(where } \lambda := y - \tfrac{c_3}{4c_4}\text{)}$$

with coefficients

$$p := \frac{1}{c_4^2}\left(c_2\,c_4 - \frac{3\,c_3^2}{8}\right), \quad q := \frac{1}{c_4^3}\left(\frac{c_3^3}{8} - \frac{c_2\,c_3\,c_4}{2} + c_1\,c_4^2\right) \quad \text{and}$$
$$r := \frac{1}{c_4^4}\left(-\frac{3c_3^4}{256} + c_4^3 c_0 - \frac{c_4^2 c_3 c_2}{4} + \frac{c_4 c_3^2 c_2}{16}\right).$$

Roots of a fourth-order polynomial

Euler's solution [3] (cont'd)

Step 2: Find roots $z_1^\star, z_2^\star, z_3^\star \in \mathbb{C}$ of Euler's resolvent cubic polynomial

$$\chi_{\text{res}}(z) := z^3 + \underbrace{2p}_{=:d_2} z^2 + \underbrace{(p^2 - 4r)}_{=:d_1} z \underbrace{-q^2}_{=:d_0} \stackrel{!}{=} 0$$

by defining

$$\tilde{q} := \frac{d_1}{3} - \frac{d_2^2}{9} \quad \text{and} \quad \tilde{r} := \frac{d_1 d_2 - 3d_0}{6} - \frac{d_2^3}{27} \quad \text{and}$$
$$s_1 := \sqrt[3]{\tilde{r} + \sqrt{\tilde{q}^3 + \tilde{r}^2}} \quad \text{and} \quad s_2 := \sqrt[3]{\tilde{r} - \sqrt{\tilde{q}^3 + \tilde{r}^2}}.$$

Roots of resolvent are then given by

$$z_1^\star = (s_1 + s_2) - \frac{d_2}{3} \quad \text{and} \quad z_{2,3}^\star = -\frac{1}{2}(s_1 + s_2) - \frac{d_2}{3} \pm \jmath \frac{\sqrt{3}}{2}(s_1 - s_2).$$

Roots of a fourth-order polynomial

Euler's solution [3] (cont'd)

Step 3: With the roots $z_1^\star, z_2^\star, z_3^\star \in \mathbb{C}$ of $\chi_{\text{res}}(z)$, the roots $\lambda_1^\star, \ldots, \lambda_4^\star \in \mathbb{C}$ of the fourth-order polynomial $\chi(\lambda)$ are given by

$$
\begin{aligned}
\lambda_1^\star &= \frac{(-1)^l}{2}\left(\sqrt{z_1^\star} + \sqrt{z_2^\star} + \sqrt{z_3^\star}\right) - \frac{c_3}{4c_4}, \\
\lambda_2^\star &= \frac{(-1)^l}{2}\left(\sqrt{z_1^\star} - \sqrt{z_2^\star} - \sqrt{z_3^\star}\right) - \frac{c_3}{4c_4}, \\
\lambda_3^\star &= \frac{(-1)^l}{2}\left(-\sqrt{z_1^\star} + \sqrt{z_2^\star} - \sqrt{z_3^\star}\right) - \frac{c_3}{4c_4}, \qquad \text{and} \\
\lambda_4^\star &= \frac{(-1)^l}{2}\left(-\sqrt{z_1^\star} - \sqrt{z_2^\star} + \sqrt{z_2^\star}\right) - \frac{c_3}{4c_4},
\end{aligned}
$$

where $l \in \{0, 1\}$ must be chosen such that [3]

$$
(-1)^l\left(\lambda_1^\star\lambda_2^\star\lambda_3^\star + \lambda_1^\star\lambda_2^\star\lambda_4^\star + \lambda_1^\star\lambda_3^\star\lambda_4^\star + \lambda_2^\star\lambda_3^\star\lambda_4^\star\right) = -q.
$$