Generic loss minimization in synchronous machines by analytically computed optimal reference currents

A unified theory considering copper+iron losses and nonlinear flux linkages

Christoph Hackl

IEEE Senior Member

Hochschule München
Fakultät für Elektrotechnik und Informationstechnik (FK04)
Labor für mechatronische und regenerative Energiesysteme (LMRES)

lmres.ee.hm.edu

30.07.2019

ALM Symposium, Schweiz

Outline

Outline

Motivation

Synchronous machines: Widely-used, compact and efficient actuators

www.bmw.de

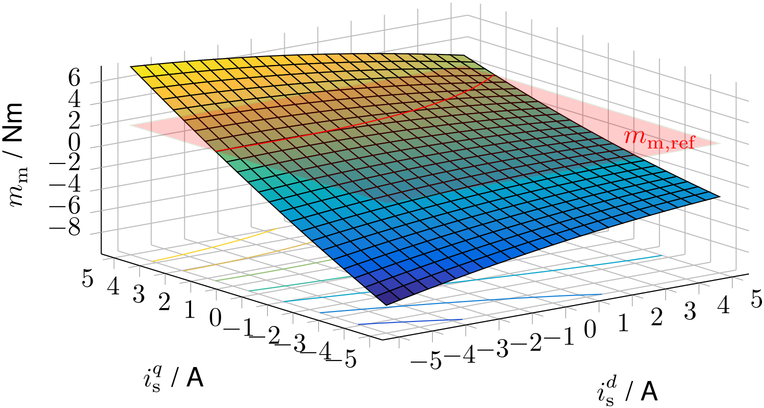
www.farmwood.co.uk

www.abb.com (from video)

- traction drives (e.g. electric vehicles or trains)
- air conditioning/fans
- pumps
- conventional power plants (e.g. hydro power plants)
- renewable energy systems (e.g. direct-drive wind turbine systems)

Motivation

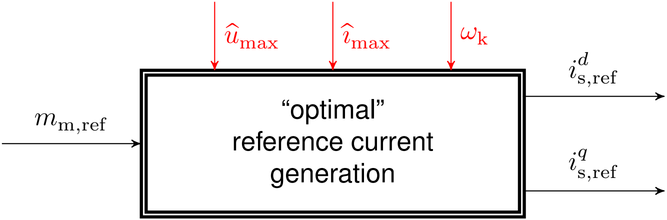
Optimal feedforward torque control problem: Anisotropic machine (with cross-coupling)



$$m_{\mathrm{m,ref}} \stackrel{!}{=} m_{\mathrm{m}}\left(i_{\mathrm{s}}^{d},\, i_{\mathrm{s}}^{q}\right) = \tfrac{3}{2}\, n_{\mathrm{p}} \Big[\underbrace{\psi_{\mathrm{pm}} i_{\mathrm{s}}^{q}}_{\text{magnetic torque}} + \underbrace{\left(L_{\mathrm{s}}^{d} - L_{\mathrm{s}}^{q}\right) i_{\mathrm{s}}^{d} i_{\mathrm{s}}^{q}}_{\text{reluctance torque}} + \underbrace{L_{\mathrm{m}}\left((i_{\mathrm{s}}^{q})^2 - (i_{\mathrm{s}}^{d})^2\right)}_{\text{cross-coupling torque}} \Big]$$

Problem statement and proposed solution

Optimal feedforward torque control problem: Optimal reference current generation

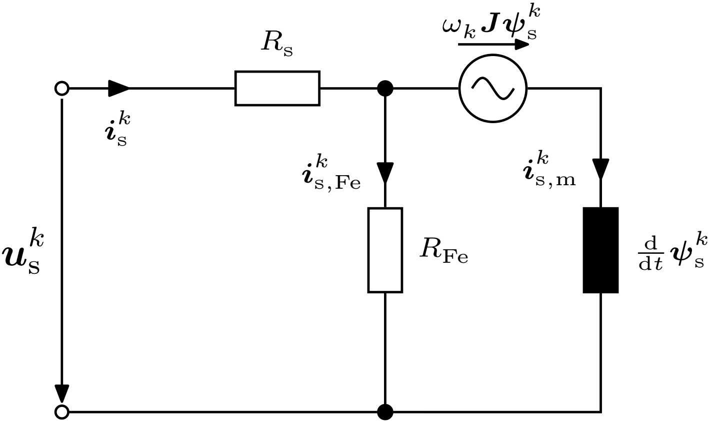


$$\boldsymbol{i}^k_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_\mathrm{k}) = \begin{pmatrix} i^d_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_\mathrm{k}) \\ i^q_{\mathrm{s,ref}}(m_{\mathrm{m,ref}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}, \omega_\mathrm{k}) \end{pmatrix}$$

- ✓ Numerical solutions and/or look-up tables
  (but: limited storage, accuracy, real-time applicability)
- **?** Analytical solutions (some do exist but impose simplifying assumptions
  such as ~~$R_\mathrm{s} = 0$, $R_\mathrm{Fe} = 0$ and/or $L_\mathrm{m} = 0$~~)

Problem statement and proposed solution

Considered anisotropic synchronous machines with iron losses [1]: Electrical circuit



Problem statement and proposed solution

Considered anisotropic synchronous machines with iron losses [1]: Modeling

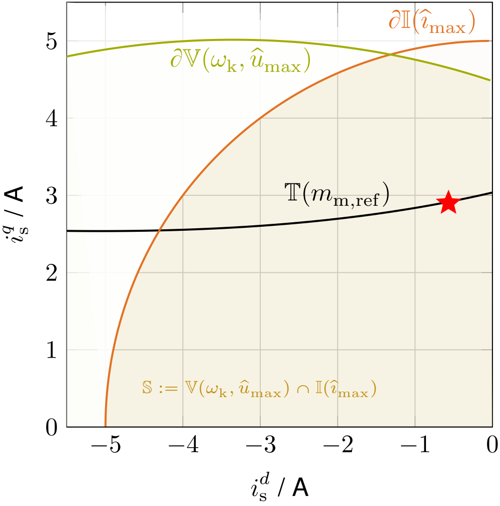
$$\begin{aligned}
\overbrace{\begin{pmatrix} u_s^d \\ u_s^q \end{pmatrix}}^{=:\boldsymbol{u}_s^k} &= R_s(\vartheta)\overbrace{\begin{pmatrix} i_s^d \\ i_s^q \end{pmatrix}}^{=:\boldsymbol{i}_s^k} + \omega_k \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{=:\boldsymbol{J}} \overbrace{\begin{pmatrix} \psi_s^d(\boldsymbol{i}_{s,m}^k) \\ \psi_s^q(\boldsymbol{i}_{s,m}^k) \end{pmatrix}}^{=:\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k)} + \cancel{\frac{d}{dt}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k)} \\
\boldsymbol{i}_s^k &= \boldsymbol{i}_{s,m}^k + \boldsymbol{i}_{s,Fe}^k = \boldsymbol{i}_{s,m}^k + \frac{\omega_k \boldsymbol{J}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k) + \cancel{\frac{d}{dt}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k)}}{R_{Fe}(\omega_k)} \\
p_{loss} &= p_{Cu}(\boldsymbol{i}_s^k) + p_{Fe}(\boldsymbol{\psi}_s^k, \omega_k) = \tfrac{3}{2} R_s \|\boldsymbol{i}_s^k\|^2 + \tfrac{3}{2}\frac{\|\omega_k \boldsymbol{J}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k) + \cancel{\frac{d}{dt}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k)}\|^2}{R_{Fe}(\omega_k)} \\
m_m(\boldsymbol{i}_{s,m}^k) &= \tfrac{3}{2} n_p \,(\boldsymbol{i}_{s,m}^k)^\top \boldsymbol{J}\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k)
\end{aligned}$$

Assumption (Affine flux linkage (at least locally))

$$\boldsymbol{\psi}_s^k(\boldsymbol{i}_{s,m}^k) \approx \underbrace{\begin{bmatrix} L_s^d & L_m \\ L_m & L_s^q \end{bmatrix}}_{=:\boldsymbol{L}_s^k \in \mathbb{R}^{2\times 2}} \boldsymbol{i}_{s,m}^k + \underbrace{\begin{pmatrix} \psi_{pm} \\ 0 \end{pmatrix}}_{=:\boldsymbol{\psi}_{pm}^k} \text{ with } \boldsymbol{L}_s^k = (\boldsymbol{L}_s^k)^\top > 0$$

Problem statement and proposed solution

Optimal reference current generation: Optimization problem(s) with multiple constraints



$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} \; -f(\boldsymbol{i}_{\mathrm{s}}^{k})$ subject to

$$\begin{cases} \|\boldsymbol{i}_{\mathrm{s}}^{k}\|^2 \leqslant \widehat{\imath}_{\max}^2, \\ \text{(current circular area)} \\ \|\boldsymbol{u}_{\mathrm{s}}^{k}(\boldsymbol{i}_{\mathrm{s}}^{k}, \omega_{\mathrm{k}})\|^2 \leqslant \widehat{u}_{\max}^2, \\ \text{(voltage elliptical area)} \\ |m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})| \leqslant |m_{\mathrm{m,ref}}|, \text{ and} \\ \mathrm{sign}(m_{\mathrm{m,ref}}) = \\ \mathrm{sign}(m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})). \end{cases}$$

e.g. $-f(\boldsymbol{i}_{\mathrm{s}}^{k}) = -\|\boldsymbol{i}_{\mathrm{s}}^{k}\|^2$ (minimize copper losses) $\Longrightarrow$ $\boldsymbol{i}_{\mathrm{s,ref}}^{k}$ at ★

Problem statement and proposed solution

Proposed solution: Implicit formulation (quadrics), optimization and intersection points

Step 1: Derivation of quadrics $Q_A(\boldsymbol{i}_\mathrm{s}^k) := (\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{A}\, \boldsymbol{i}_\mathrm{s}^k + 2\boldsymbol{a}^\top \boldsymbol{i}_\mathrm{s}^k + \alpha$:

- Current circular area: $(i_\mathrm{s}^d)^2 + (i_\mathrm{s}^q)^2 \leqslant \widehat{\imath}_\mathrm{max}^2 \iff \|\boldsymbol{i}_\mathrm{s}^k\|^2 = (\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{I}_2 \boldsymbol{i}_\mathrm{s}^k \leqslant \widehat{\imath}_\mathrm{max}^2$

$$\Longrightarrow \quad \mathbb{I}(\widehat{\imath}_\mathrm{max}) := \left\{\, \boldsymbol{i}_\mathrm{s}^k \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{I}_2 \boldsymbol{i}_\mathrm{s}^k - \widehat{\imath}_\mathrm{max}^2 \leqslant 0 \,\right\}$$

- Voltage elliptical area: $(u_\mathrm{s}^d)^2 + (u_\mathrm{s}^q)^2 \leqslant \widehat{u}_\mathrm{max}^2 \iff \|\boldsymbol{u}_\mathrm{s}^k(\boldsymbol{i}_\mathrm{s}^k, \omega_\mathrm{k})\|^2 = \ldots \leqslant \widehat{u}_\mathrm{max}^2$

$$\Longrightarrow \quad \mathbb{V}(\omega_\mathrm{k}, \widehat{u}_\mathrm{max}) := \left\{\, \boldsymbol{i}_\mathrm{s}^k \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{V}(\omega_\mathrm{k})\, \boldsymbol{i}_\mathrm{s}^k + 2\, \boldsymbol{v}(\omega_\mathrm{k})^\top \boldsymbol{i}_\mathrm{s}^k + \nu(\omega_\mathrm{k}, \widehat{u}_\mathrm{max}) \leqslant 0 \right\}$$

- Reference torque hyperbola: $m_\mathrm{m}(\boldsymbol{i}_\mathrm{s,m}^k) = \frac{3}{2} n_\mathrm{p} (\boldsymbol{i}_\mathrm{s,m}^k)^\top \boldsymbol{J} \boldsymbol{\psi}_\mathrm{s}^k(\boldsymbol{i}_\mathrm{s,m}^k) \stackrel{!}{=} m_\mathrm{m,ref} \iff \ldots$

$$\Longrightarrow \quad \mathbb{T}(m_\mathrm{m,ref}) := \left\{\, \boldsymbol{i}_\mathrm{s}^k \in \mathbb{R}^2 \;\middle|\; (\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{T} \boldsymbol{i}_\mathrm{s}^k + 2\, \boldsymbol{t}^\top \boldsymbol{i}_\mathrm{s}^k - m_\mathrm{m,ref} = 0 \,\right\}$$

- ...

Problem statement and proposed solution

Proposed solution: Implicit formulation (quadrics), optimization and intersection points

Step 2: Optimization problem with equality constraint

$$\max_{\boldsymbol{i}_\mathrm{s}^k} -\Big(\underbrace{(\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{A} \boldsymbol{i}_\mathrm{s}^k + 2\boldsymbol{a}^\top \boldsymbol{i}_\mathrm{s}^k + \alpha}_{=:Q_A(\boldsymbol{i}_\mathrm{s}^k)} \Big) \text{ s.t. } \underbrace{(\boldsymbol{i}_\mathrm{s}^k)^\top \boldsymbol{B} \boldsymbol{i}_\mathrm{s}^k + 2\boldsymbol{b}^\top \boldsymbol{i}_\mathrm{s}^k + \beta}_{=:Q_B(\boldsymbol{i}_\mathrm{s}^k)} = 0$$

$\Longrightarrow$ Hyperbolas for e.g. MTPC, MTPL or MTPV (with $R_\mathrm{s}, R_\mathrm{Fe} \neq 0$ & $L_\mathrm{m} \neq 0$)

Step 3: Intersection of two quadrics (e.g. voltage ellipse and current circle)

$$\boldsymbol{i}_{\mathrm{s,ref}}^k := \arg\min_{\|\boldsymbol{i}_\mathrm{s}^k\|} \Big\{ \boldsymbol{i}_\mathrm{s}^k \in \mathbb{R}^2 \,\Big|\, Q_A(\boldsymbol{i}_\mathrm{s}^k) = 0 \;\wedge\; Q_B(\boldsymbol{i}_\mathrm{s}^k) = 0 \Big\}$$

$\Longrightarrow$ Optimal operation point ★ (reference current)

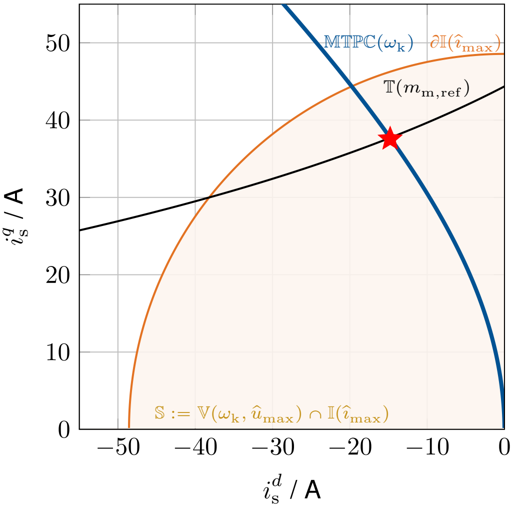
Both lead to subproblem of solving a fourth-order (quartic) polynomial

$$\chi(\lambda) := c_4\,\lambda^4 + c_3\,\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 \stackrel{!}{=} 0$$

$\Longrightarrow$ Analytical solutions exist (e.g. Euler's solution, see [2])

Optimal operation strategies

Maximum Torque per Current (**MTPC @** $0 \cdot \omega_{\mathrm{m,rat}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{k}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{k})^{\top} \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{k} + 2\,\boldsymbol{t}^{\top} \boldsymbol{i}_{\mathrm{s}}^{k}}_{= m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})} - m_{\mathrm{m,ref}} = 0$$

Solution set

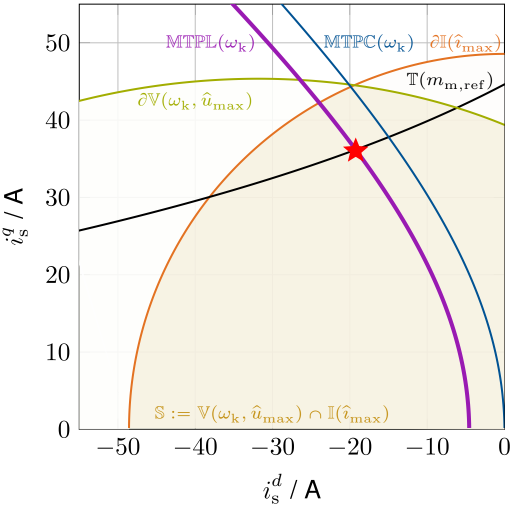
$$\mathbb{MTPC}(\omega_{\mathrm{k}}) := \left\{ \boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{R}^2 \,\middle|\, (\boldsymbol{i}_{\mathrm{s}}^{k})^{\top} \boldsymbol{M}_{\mathrm{C}}\, \boldsymbol{i}_{\mathrm{s}}^{k} + 2\,\boldsymbol{m}_{\mathrm{C}}^{\top} \boldsymbol{i}_{\mathrm{s}}^{k} = 0 \right\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{k} = \mathbb{MTPC} \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Maximum Torque per Losses (**MTPL** @ $1 \cdot \omega_{\mathrm{m,rat}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{k}) - p_{\mathrm{Fe}}(\boldsymbol{i}_{\mathrm{s}}^{k}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{k})^{\top} \boldsymbol{T} \boldsymbol{i}_{\mathrm{s}}^{k} + 2\, \boldsymbol{t}^{\top} \boldsymbol{i}_{\mathrm{s}}^{k}}_{= m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})} - m_{\mathrm{m,ref}} = 0$$

Solution set

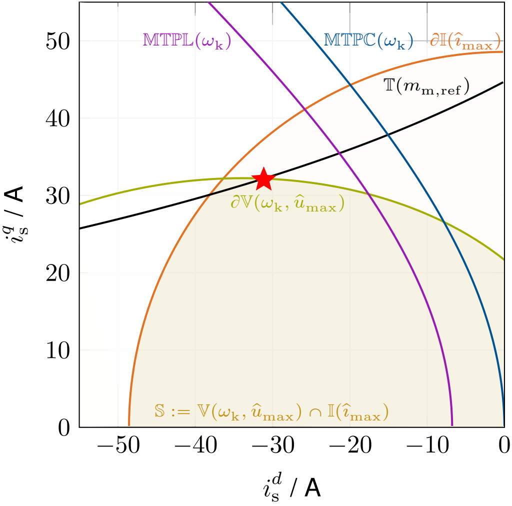
$$\mathbb{MTPL}(\omega_{\mathrm{k}}) := \Big\{ \boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{R}^2 \;\Big|\; (\boldsymbol{i}_{\mathrm{s}}^{k})^{\top} \boldsymbol{M}_{\mathrm{L}}\, \boldsymbol{i}_{\mathrm{s}}^{k} + 2\, \boldsymbol{m}_{\mathrm{L}}^{\top} \boldsymbol{i}_{\mathrm{s}}^{k} + \mu_{\mathrm{L}} = 0 \Big\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{k} = \mathbb{MTPL} \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Field Weakening (**FW** @ $1.40 \cdot \omega_{\mathrm{m,rat}}$): Reference torque feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} -p_{\mathrm{Cu}}(\boldsymbol{i}_{\mathrm{s}}^{k}) - p_{\mathrm{Fe}}(\boldsymbol{i}_{\mathrm{s}}^{k}) \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{k})^{\top}\boldsymbol{T}\boldsymbol{i}_{\mathrm{s}}^{k} + 2\,\boldsymbol{t}^{\top}\boldsymbol{i}_{\mathrm{s}}^{k}}_{=m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})} - m_{\mathrm{m,ref}} = 0$$

Feasible set

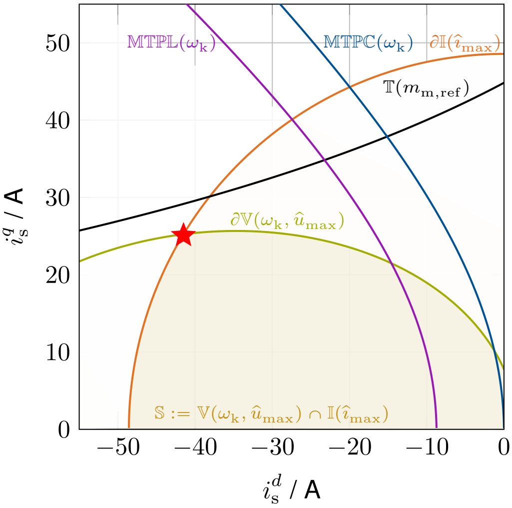
$$\mathbb{FW}(m_{\mathrm{m,ref}}, \omega_{\mathrm{k}}, \widehat{u}_{\max}) := \mathbb{S}(\omega_{\mathrm{k}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}) \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{k} = \partial\mathbb{V}(\omega_{\mathrm{k}}, \widehat{u}_{\max}) \cap \mathbb{T}(m_{\mathrm{m,ref}})$$

Optimal operation strategies

Maximum Current (**MC** @ $1.75 \cdot \omega_{\mathrm{m,rat}}$): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})$$

Feasible set

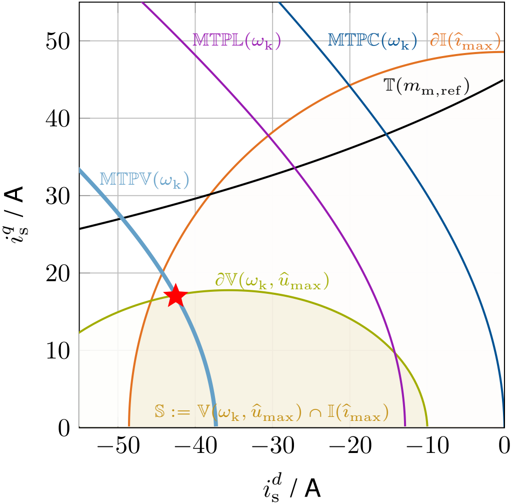
$$\mathbb{MC}(\omega_{\mathrm{k}}, \widehat{u}_{\max}, \widehat{\imath}_{\max}) := \mathbb{V}(\omega_{\mathrm{k}}, \widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Optimal current reference (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{k} = \partial\mathbb{V}(\omega_{\mathrm{k}}, \widehat{u}_{\max}) \cap \partial\mathbb{I}(\widehat{\imath}_{\max})$$

Optimal operation strategies

Maximum Torque per Voltage (**MTPV @** $2.50 \cdot \omega_{\mathrm{m,rat}}$): Reference torque **not** feasible



Optimization problem

$$\max_{\boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{S}} -\|\boldsymbol{u}_{\mathrm{s}}^{k}(\boldsymbol{i}_{\mathrm{s}}^{k})\|^{2} \quad \text{s.t.}$$

$$\underbrace{(\boldsymbol{i}_{\mathrm{s}}^{k})^{\top}\boldsymbol{T}\boldsymbol{i}_{\mathrm{s}}^{k} + 2\,\boldsymbol{t}^{\top}\boldsymbol{i}_{\mathrm{s}}^{k}}_{=m_{\mathrm{m}}(\boldsymbol{i}_{\mathrm{s}}^{k})} - m_{\mathrm{m,ref}} = 0$$

Solution set

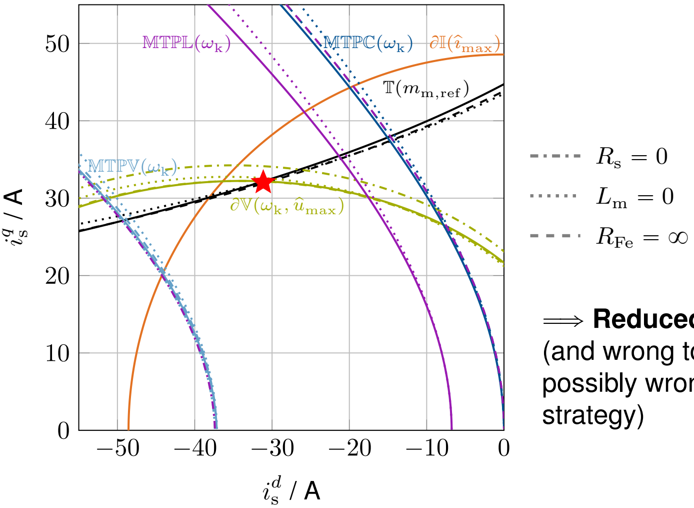
$$\mathbb{MTPV}(\omega_{\mathrm{k}}) := \Big\{\, \boldsymbol{i}_{\mathrm{s}}^{k} \in \mathbb{R}^{2} \;\Big|\; (\boldsymbol{i}_{\mathrm{s}}^{k})^{\top}\boldsymbol{M}_{\mathrm{V}}\,\boldsymbol{i}_{\mathrm{s}}^{k} + 2\,\boldsymbol{m}_{\mathrm{V}}^{\top}\boldsymbol{i}_{\mathrm{s}}^{k} + \mu_{\mathrm{V}} = 0 \Big\}$$

Optimal reference currents (★)

$$\boldsymbol{i}_{\mathrm{s,ref}}^{k} = \mathbb{MTPV}(\omega_{\mathrm{k}}) \cap \partial\mathbb{V}(\omega_{\mathrm{k}}, \hat{u}_{\max})$$

Optimal operation strategies

Field Weakening (**FW** @ $1.40 \cdot \omega_{\mathrm{m,rat}}$): Illustration of the effect of neglecting R_{s}, R_{Fe} and/or L_{m}



$\Longrightarrow$ **Reduced efficiency**! (and wrong torque and/or possibly wrong operation strategy)

Implementation results

Laboratory setup

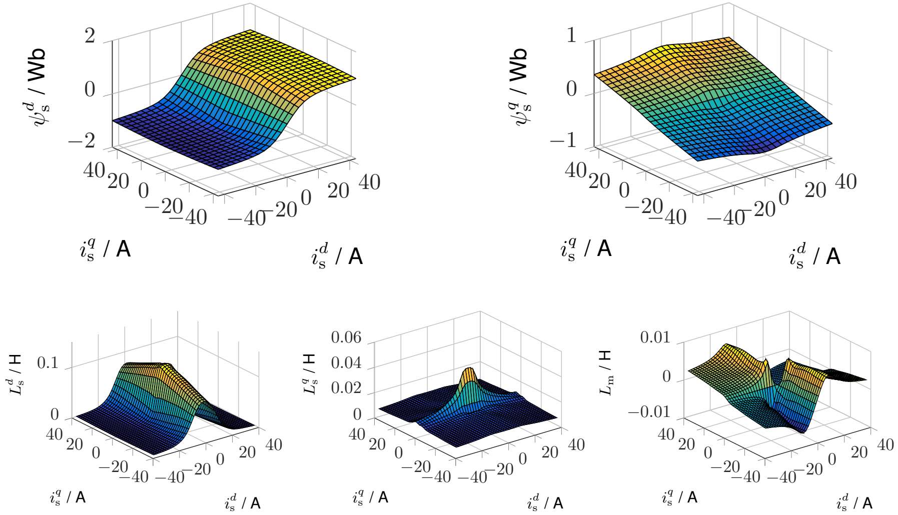
Reluctance SM (RSM) and Permanent-magnet SM

Host PC (rapid prototyping)

Real-time system and VSIs

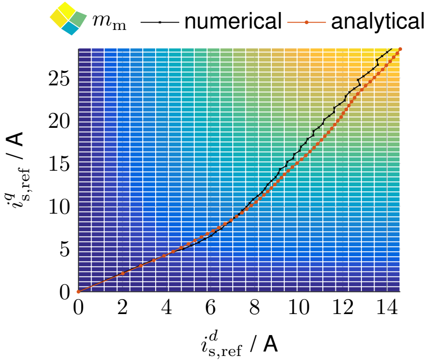
Implementation results

RSM: Nonlinear flux linkages and differential inductances

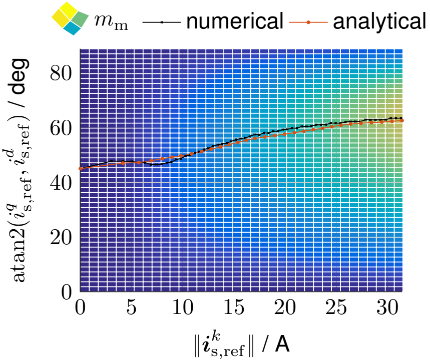


Implementation results

RSM: Comparison of numerical and analytical solutions for MTPC



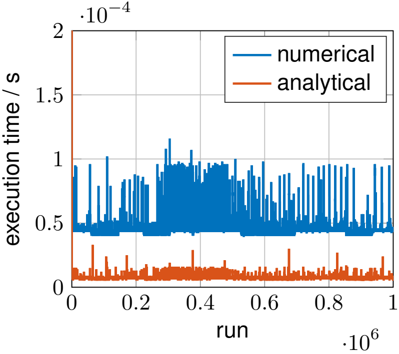
Cartesian coordinates

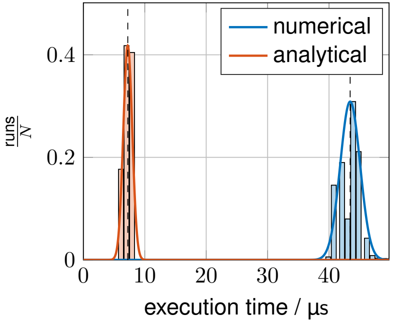


polar coordinates

Implementation results

RSM: Comparison of computational load

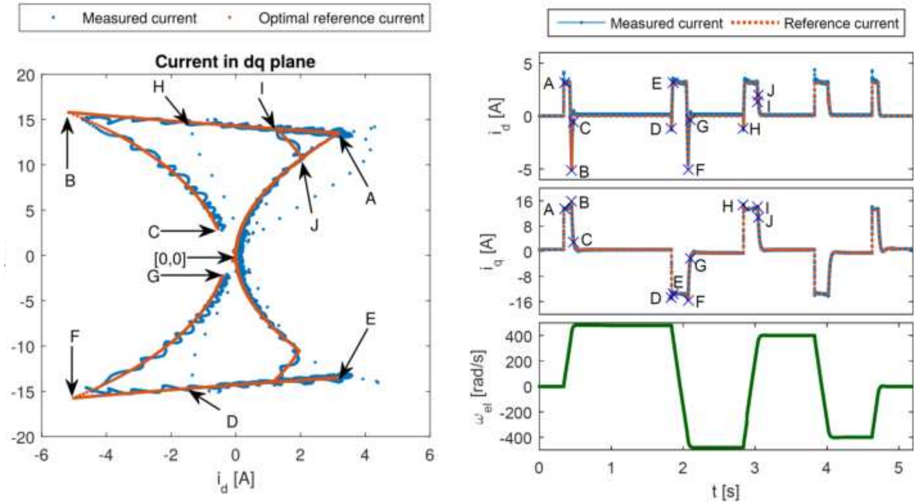




- Execution times for $N = 10^6$ runs (downsampled by factor 100)
- Standard deviations $\sigma_n = 1{,}61 \cdot 10^{-6}$ s and $\sigma_a = 0{,}73 \cdot 10^{-6}$ s
- Average execution times $\mu_n = 43{,}4 \cdot 10^{-6}$ s and $\mu_a = 7{,}23 \cdot 10^{-6}$ s ($>$**6x faster**)

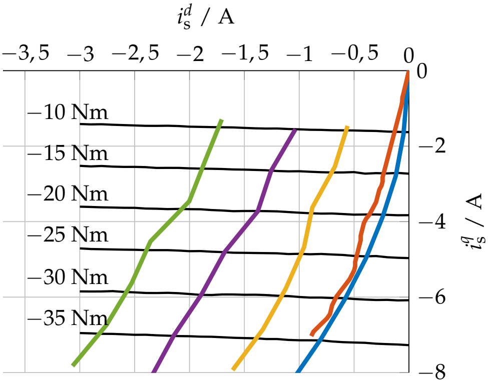
Implementation results

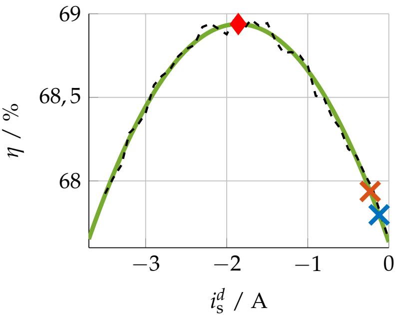
PME-RSM: Implementation by colleagues ($p_{\mathrm{m,rat}} = 4{,}5\,\mathrm{kW}$ & $L_{\mathrm{s}}^{d}/L_{\mathrm{s}}^{q} = 1.43$) [3]



Implementation results

IPMSM (generator mode): Efficiency enhancements considering iron losses [4]





Conclusion

Summary and future work

To take home

- **unified** theory for MTPC, MTPL, FW, MC and MTPV based on [5]
  - **quadrics** (implicit formulation),
  - **optimization problems** with **equality constraint**, and
  - **intersection** of two quadrics
- **analytical** solution for reference currents (& transition points)
- $R_{\mathrm{s}} \neq 0$, $R_{\mathrm{Fe}} \neq 0$ and $L_{\mathrm{m}} \neq 0$ (can all be considered **simultaneously**)
- **fast(er)** and **more accurate** computation
- **applicable** in real-world (e.g. nonlinear RSM, PME-RSM or IPMSM; also IMs)

Future work

- convergence (mathematical proof for nonlinear flux linkages)
- more extensive experimental validation
- impact of (parameter) uncertainties
- consideration of *rotor* iron losses (e.g. for induction machines)
- combination with minimization of *conduction and switching losses*

References

Journal article (published 2017)

INTERNATIONAL JOURNAL OF CONTROL, 2017
https://doi.org/10.1080/00207179.2017.1338359

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

A unified theory for optimal feedforward torque control of anisotropic synchronous machines

Hisham Eldeeb[a], Christoph M. Hackl[a,*], Lorenz Horlbeck[b] and Julian Kullick[a]

[a]Research Group 'Control of Renewable Energy Systems (CRES)', Munich School of Engineering (MSE), Technical University of Munich (TUM), Munich, Germany; [b]Institute of Automotive Technology (FTM), Technical University of Munich (TUM), Munich, Germany

ABSTRACT
A unified theory for optimal feedforward torque control of anisotropic synchronous machines with *non-negligible stator resistance* and *mutual inductance* is presented which allows to *analytically* compute (1) the optimal direct and quadrature reference currents for all operating strategies, such as maximum torque per current (MTPC), maximum current, field weakening, maximum torque per voltage (MTPV) or maximum torque per flux (MTPF), and (2) the transition points indicating when to switch between the operating strategies due to speed, voltage or current constraints. The analytical solutions allow for an (almost) instantaneous selection and computation of actual operation strategy and corresponding reference currents. Numerical methods (approximating these solutions only) are *no longer* required. The unified theory is based on one simple idea: all optimisation problems, their respective constraints and the computation of the intersection point(s) of voltage ellipse, current circle or torque, MTPC, MTPV, MTPF hyperbolas are reformulated implicitly as *quadrics* which allows to invoke the Lagrangian formalism and to find the roots of fourth-order polynomials analytically. The proposed theory is suitable for any anisotropic synchronous machine. Implementation and measurement results illustrate effectiveness and applicability of the theoretical findings in real world.

ARTICLE HISTORY
Received 18 March 2017
Accepted 31 May 2017

KEYWORDS
Maximum torque per ampere (MTPA); maximum torque per current (MTPC); maximum torque per voltage (MTPV); maximum torque per flux (MTPF); maximum current (MC); field weakening (FW); analytical solution; optimal feedforward torque control; efficiency; copper losses; anisotropy; synchronous machine; interior

References I

[1] L. Xu and J. Yao. A compensated vector control scheme of a synchronous reluctance motor including saturation and iron losses. *IEEE Transactions on Industry Applications*, 28(6):1330–1338, 1992.

[2] R.W.D Nickalls. The quartic equation: Invariants and Euler's solution revealed. *Mathematical Gazette*, 93(526):66–75, 2009.

[3] Antonín Glac, Václav Šmídl, and Zdenek Peroutka. Optimal feedforward torque control of synchronous machines with time-varying parameters. In *Proceedings of the 44th Annual Conference of the IEEE Industrial Electronics Society (IECON)*, pages 613–618, 2018.

[4] Simon Krüner and Christoph M. Hackl. Experimental identification of the optimal current vectors for a permanent-magnet synchronous machine in wave energy converters. *Energies*, 12(862), 2019.

[5] Hisham Eldeeb, Christoph M. Hackl, Lorenz Horlbeck, and Julian Kullick. A unified theory for optimal feedforward torque control of anisotropic synchronous machines. *International Journal of Control*, 2017.

[6] Shigeo Morimoto. Trend of permanent magnet synchronous machines. *IEEJ Transactions on Electrical and Electronic Engineering*, 2(2):101–108, mar 2007.