Is multiple-objective model-predictive control “optimal”?

Christoph Hackl
(joint work with: F. Larcher, A. Dötlinger and R. Kennel)

Technische Universität München
Institute for Electrical Drive Systems and Power Electronics

SLED/PRECEDE United Symposium 2013, Munich
17th October 2013

Model-predictive control: Problem statement

- **Optimality** with respect to performance index

$$J = \sum_{i=1}^{n} w_i J_i, \qquad n \in \mathbb{N},\ \forall i \in \{1, \ldots, n\}: \quad J_i, w_i \geqslant 0. \tag{COST}$$

- but **not** necessarily “optimal” with respect to some J_i
  - How to choose weighting factors $w_1, \ldots, w_n$?
  - What is optimal for the human operator (“human performance index”)?
  - Does (COST) reflect the “human performance index” adequately?
- Intuition: “Increase w_i to “optimize” sub-cost J_i”
  $\rightsquigarrow$ We will see: This does **not** hold in general!

Contents

Model-predictive control: Considered system

Linear time-invariant (discrete) system

$$\left.\begin{array}{rcl} \boldsymbol{x}[k+1] &=& \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k], \qquad \boldsymbol{x}[0] = \boldsymbol{x}_0 \in \mathbb{R}^n \\ y[k] &=& \boldsymbol{c}^\top \boldsymbol{x}[k] \end{array}\right\} \tag{SYS}$$

Standard assumption [1, p. 48] on system (SYS):

(A_1) controllability, i.e. $\det[\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b}] \neq 0$.

(A_2) full state feedback, i.e. $\boldsymbol{x}[k]$ is available for all k.

($A_2^\star$) observability, i.e. $\det \begin{bmatrix} \boldsymbol{c}^\top \\ \boldsymbol{c}^\top \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^\top \boldsymbol{A}^{n-1} \end{bmatrix} \neq 0$ (observer design).

(A_3) model-based prediction, i.e. $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are known.

Model-predictive control: Problem & Solution

Problem: Find

$$\boldsymbol{u}_\star[k+1] := \big(u_\star[k+1], \ldots, u_\star[k+n_P-1]\big)^\top \in \mathbb{R}^{n_P-1}, \tag{1}$$

to

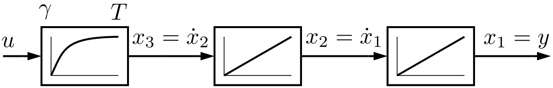
$$\left.\begin{aligned}
&\min_{\boldsymbol{v}} J_0\big(k,u,\boldsymbol{x},\boldsymbol{v}\big) = \min_{\boldsymbol{v}} \Big(\boldsymbol{g}\big(k,u,\boldsymbol{x}\big)^\top \boldsymbol{v} + \tfrac{1}{2}\boldsymbol{v}^\top \boldsymbol{H}\big(k,u,\boldsymbol{x}\big)\boldsymbol{v}\Big), \\
&\text{where } \forall (k,u,\boldsymbol{x}) \in \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R}^n: \\
&\text{(i) } 0 < \boldsymbol{H}(k,u,\boldsymbol{x}) = \boldsymbol{H}(k,u,\boldsymbol{x})^\top \in \mathbb{R}^{(n_P-1)\times(n_P-1)} \\
&\text{(ii) } \boldsymbol{H}(k,u,\boldsymbol{x})^{-1}\boldsymbol{g}(k,u,\boldsymbol{x}) = \boldsymbol{0}_{n_P} \Leftrightarrow \boldsymbol{g}(k,u,\boldsymbol{x}) = \boldsymbol{0}_{n_P-1}.
\end{aligned}\right\} \tag{2}$$

Solution: (analytical, unique and globally optimal)

$$\boldsymbol{u}_\star[k+1] = \operatorname*{argmin}_{\boldsymbol{v}} J_0\big(k,u,\boldsymbol{x},\boldsymbol{v}\big) = -\boldsymbol{H}(k,u[k],\boldsymbol{x}[k])^{-1}\boldsymbol{g}(k,u[k],\boldsymbol{x}[k]). \tag{3}$$

Example: System model

Continuous time block diagram

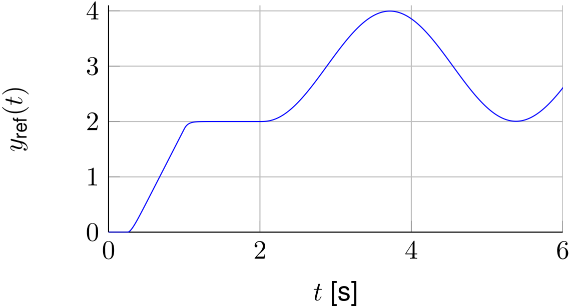


Discrete system model

$$\left.\begin{aligned} \boldsymbol{x}[k+1] &= \underbrace{\begin{bmatrix} 1 & T_S & 0 \\ 0 & 1 & T_S \\ 0 & 0 & 1-\frac{T_S}{T} \end{bmatrix}}_{=:\boldsymbol{A}\in\mathbb{R}^{3\times 3}} \boldsymbol{x}[k] + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{\gamma T_S}{T} \end{pmatrix}}_{=:\boldsymbol{b}\in\mathbb{R}^{3}} u[k], \\ y[k] &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{=:\boldsymbol{c}^\top\in\mathbb{R}^{1\times 3}} \boldsymbol{x}[k], \quad \boldsymbol{x}[0] = \boldsymbol{x}_0 \in \mathbb{R}^3, \end{aligned}\right\} \tag{4}$$

- controllable and observable, since $\det[\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}] \neq \det\begin{bmatrix} \boldsymbol{c}^\top \\ \boldsymbol{c}^\top\boldsymbol{A} \\ \boldsymbol{c}^\top\boldsymbol{A}^2 \end{bmatrix} \neq 0$.
- full state feedback, i.e. $\boldsymbol{x}[k]$ is available for all k.
- $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are known.

Example: Tracking problem



Performance index

$$J(k, u[k], \boldsymbol{x}[k], \boldsymbol{u}_\star[k+1]) = \underbrace{\widehat{\boldsymbol{e}}[k+1]^\top \boldsymbol{W}_1 \widehat{\boldsymbol{e}}[k+1]}_{=:J_1(k,u[k],\boldsymbol{x}[k],\boldsymbol{u}_\star[k+1])} + \underbrace{\boldsymbol{u}_\star[k+1]^\top \boldsymbol{W}_2 \boldsymbol{u}_\star[k+1]}_{=:J_2(\boldsymbol{u}_\star[k+1])} \tag{5}$$

where $\boldsymbol{W}_1 \geqslant 0$, $\boldsymbol{W}_2 > 0$ and $\widehat{\boldsymbol{e}}[k+1] = \boldsymbol{y}_{\mathrm{ref}}[k+1] - \widehat{\boldsymbol{y}}[k+1]$.

Example: Re-Formulation as standard MPC

Note that

$$\boldsymbol{W}_1 \geqslant 0 \qquad \text{and} \qquad \boldsymbol{W}_2 > 0$$

imply (see Section II.E in paper)

$$\begin{aligned} J(k, u[k], \boldsymbol{x}[k], \boldsymbol{u}_\star[k+1]) &= \cancel{f(k, u[k], \boldsymbol{x}[k])} \\ &\quad + \underbrace{\boldsymbol{g}(k, u[k], \boldsymbol{x}[k])^\top \boldsymbol{u}_\star[k+1] + \frac{1}{2}\boldsymbol{u}_\star[k+1]^\top \boldsymbol{H} \boldsymbol{u}_\star[k+1]}_{=: J_0(k, u[k], \boldsymbol{x}[k], \boldsymbol{u}_\star[k+1])} \qquad (6) \end{aligned}$$

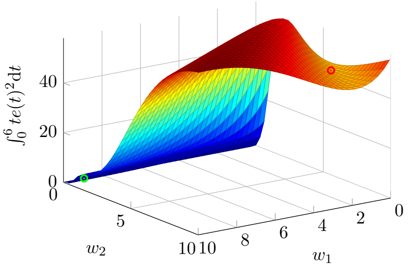
where $\boxed{\boldsymbol{H} := 2\big(\boldsymbol{G}^\top \boldsymbol{W}_1 \boldsymbol{G} + \boldsymbol{W}_2\big) > 0}$ with

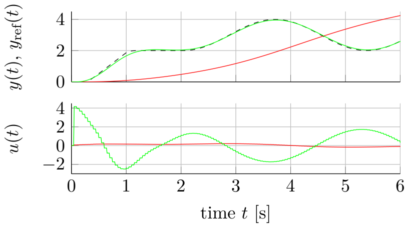
$$\boldsymbol{G} := \begin{bmatrix} \boldsymbol{c}^\top \boldsymbol{b} & 0 & \cdots & 0 \\ \boldsymbol{c}^\top \boldsymbol{A}\boldsymbol{b} & \boldsymbol{c}^\top \boldsymbol{b} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \boldsymbol{c}^\top \boldsymbol{A}^{n_P-2}\boldsymbol{b} & \boldsymbol{c}^\top \boldsymbol{A}^{n_P-3}\boldsymbol{b} & \cdots & \boldsymbol{c}^\top \boldsymbol{b} \end{bmatrix} \in \mathbb{R}^{(n_P-1)\times(n_P-1)}.$$

Example: Human performance index

Integral time square error (ITSE) criterion

$$\text{ITSE}(e(\cdot)) := \int_0^{t_{\text{end}}} \tau e(\tau)^2 \, \mathrm{d}\tau \approx \sum_{k=0}^{t_{\text{end}}/T_S} k T_S^2 e[k]^2$$



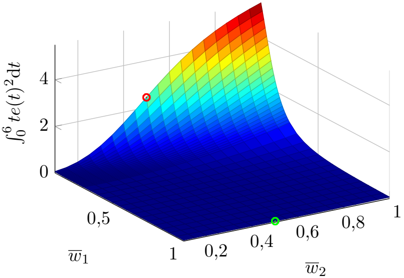


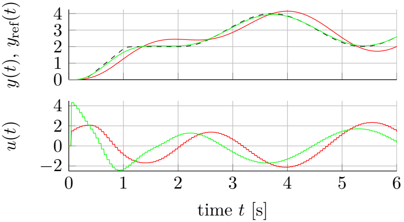
- ● **bad** weighting factor design
- ● **good** weighting factor design
- $\Longrightarrow$ How to achieve a good design?

Example: Scaled performance index (see [1, Sec. 6])

$$\overline{J} := \frac{\overbrace{\widehat{\boldsymbol{e}}[k+1]^\top \boldsymbol{W}_1 \widehat{\boldsymbol{e}}[k+1]}^{=: J_1}}{J_{1,\max}} + \frac{\boldsymbol{u}_\star[k+1]^\top \boldsymbol{W}_2 \boldsymbol{u}_\star[k+1]}{J_{2,\max}}$$

with $\boldsymbol{W}_1 = \operatorname{diag}\{\overline{w}_1, \ldots, \overline{w}_1\} > 0,\ \boldsymbol{W}_2 = \operatorname{diag}\{\overline{w}_2, \ldots, \overline{w}_2\} > 0$





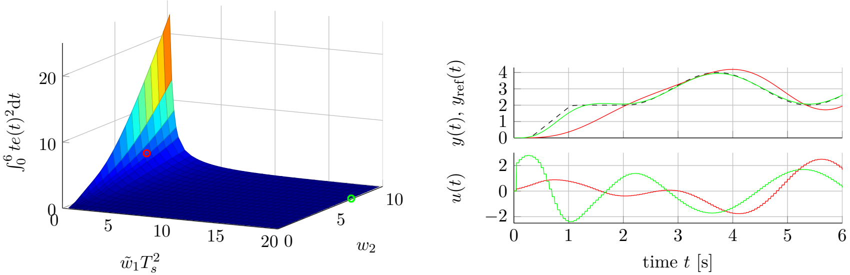
- ● **bad** / ● **good** weighting factor design
- $\Longrightarrow$ Intuition holds again: “Increase $\overline{w}_1$ to “optimize” sub-cost J_1!”

Example: Time-varying weighting matrix (similar to [2])

$$J(k, u[k], \boldsymbol{x}[k], \boldsymbol{u}_\star[k+1]) =$$

$$\underbrace{\widehat{\boldsymbol{e}}[k+1]^\top \color{red}{\boldsymbol{W}_1[k]}\color{black}\widehat{\boldsymbol{e}}[k+1]}^{=:J_1 \approx \text{ITSE}(\hat{e}(\cdot))|_{[k,\dots,k-n_P]} = \int_{kT_S}^{(k+n_P)T_S} \tau e(\tau)^2 \, \mathrm{d}\tau} \qquad + \boldsymbol{u}_\star[k+1]^\top \boldsymbol{W}_2 \boldsymbol{u}_\star[k+1]$$

with $\boldsymbol{W}_1[k] = \tilde{w}_1 T_S^2 \,\mathrm{diag}\{k, \dots k+n_P\} \geqslant 0, \; \boldsymbol{W}_2 = \mathrm{diag}\{w_2, \dots, w_2\} > 0$



- ● **bad** / ● **good** weighting factor design
- $\Longrightarrow$ Intuition holds again: "Increase $\tilde{w}_1$ to "optimize" sub-cost J_1!"

References

[1] B. D. Anderson and J. B. Moore. *Optimal Control: Linear Quadratic Methods*. Precentice Hall Information and System Sciences Series. Prentice Hall International Inc., Englewood Cliffs, New Jersey, 1990.

[2] D. Chen, L. Bako, and S. Lecoeuche. The minimum-time problem for discrete-time linear systems: A non-smooth optimization approach. In *Proceedings of the IEEE International Conference on Control Applications*, pages 196–201, 2012. ISBN 9781467345040.